18.700 - Fall 2006 - Solutions to Problem Set 6

Problem 1.

Suppose that A is diagonalizable. Then it has a basis of eigenvectors $\mathcal{B} = \{v_1, \ldots, v_n\}$. The rank of A is exactly the number of eigenvectors corresponding to a nonzero eigenvalue. Suppose $Av_i = 0$ for $i = 1, \ldots, n-1$ and $Av_n = \alpha v_n$ with $\alpha \neq 0$. The matrix $M = M_{\mathcal{B}}^{\mathcal{B}}(A)$ is diagonal, with $M_{ii} = 0$ for $i = 1, \ldots, n-1$ and $M_{nn} = \alpha$. Then $\operatorname{tr}(A) = \operatorname{tr}(M) = \alpha \neq 0$.

Vice versa, suppose that $\operatorname{tr}(A) \neq 0$.
Because $\operatorname{rk}(A) = 1$, we have $\dim\ker(A) = n-1$. Let $\{v_1, \ldots, v_{n-1}\}$ be a basis of $\ker(A)$ and pick $v_n \in \mathbb{F}^n \setminus \ker(A)$, so that $\mathcal{B} = \{v_1, \ldots, v_n\}$ is a basis of $\mathbb{F}^n$. The matrix $M = \mathcal{M}_{\mathcal{B}}^{\mathcal{B}}(A)$ looks like

$$M = \begin{pmatrix} 0 & \ldots & 0 & * \\ 0 & \ldots & 0 & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & * \\ 0 & \ldots & 0 & \alpha \end{pmatrix}$$

and $\operatorname{tr}(M) = \operatorname{tr}(A) = \alpha \neq 0$. Moreover, the characteristic polynomial $p_A(t) = p_M(t) = \det(M - tI) = (\alpha - t)(-t)^{n-1}$, thus there is an eigenvector $0 \neq w \in \mathbb{F}^n$ for A corresponding to the eigenvalue α. Clearly, $\mathcal{C} = \{v_1, \ldots, v_{n-1}, w\}$ is a basis of $\mathbb{F}^n$ of eigenvectors of A. Hence, A is diagonalizable.

Problem 2.

(a) The matrix $A_c - tI$ is lower triangular, so that its characteristic polynomial is $p_{A_c}(t) = -t(2-t)(3-t)(c-t)$.

(b) The eigenvalues of A_c are $\{0, 2, 3, c\}$.
We always have $E_0 = \operatorname{span}\{e_4\}$, $E_2 = \operatorname{span}\{(2c-4)e_1 - 2e_2 + 2e_3 + e_4\}$, $E_3 = \operatorname{span}\{3e_3 + e_4\}$.
Moreover, if $c \neq 0, 2, 3$, then we also have $E_c = \operatorname{span}\{(c-3)e_2 + e_3 + c^{-1}e_4\}$.

(c) The sum of the multiplicities of the eigenvalues is 4 if $c \neq 0, 2, 3$ and is 3 otherwise. Thus, A_c is diagonalizable if and only if $c \neq 0, 2, 3$.

(d) Yes, for $c = 2$. In this case, we show that there are bases $\mathcal{B} = \{v_1, v_2, v_3, v_4\}$ and $\mathcal{C} = \{w_1, w_2, w_3, w_4\}$ such that

$$M_{\mathcal{B}}^{\mathcal{B}}(A_2) = M_{\mathcal{C}}^{\mathcal{C}}(B) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (*)$$

so that A_2 and B are similar.
Define $v_4 = w_4 = e_4$ corresponding to the kernel (of A_2 and B respectively).
Define $v_3 = e_3 + e_4/3$ and $w_3 = e_3$, corresponding to the eigenspace of eigenvalue 3.

Define $v_2 = -2e_2 + 2e_3 + e_4$ and $w_2 = -e_2 + e_3$, corresponding to the eigenspace of eigenvalue 2.
Finally, define $v_1 = -2e_1 + e_2 + e_3$ and $w_1 = v_1/2$, so that $\ker(A_2 - 2_I)^2 = \operatorname{span}\{v_1, v_2\}$ and $\ker(B - 2I)^2 = \operatorname{span}\{w_1, w_2\}$.
You have to check that, with the bases defined above, (*) holds.

(e) Direct computation shows that the characteristic polynomial of $B - B^t$ is $p(\lambda) = \lambda^2(\lambda^2 + 2)$, which is not a product of factors of degree 1 in $\mathbb{Q}[\lambda]$. So $B - B^t$ is not similar to an upper triangular matrix in $\mathcal{M}_{4\times 4}(\mathbb{Q})$.

Problem 3.

(a) Let B be the invertible minor of A made of the columns $C_{i_1}, \dots, C_{i_k}$ and the rows $R_{j_1}, \dots, R_{j_k}$. Consider the subsets of linearly independent vectors $\mathcal{B} = \{e_{i_1}, \dots, e_{i_k}\}$ and $\mathcal{C} = \{e_{j_1}, \dots, e_{j_k}\}$. Define vector subspaces $V = \operatorname{span}(\mathcal{B}) \subseteq \mathbb{F}^n$ and $W = \operatorname{span}(\mathcal{C}) \subseteq \mathbb{F}^n$. Notice that there is an inclusion $f : V \to \mathbb{F}^n$, and a (surjective) homomorphism $g : \mathbb{F}^n \to W$ such that $g(e_h) = e_h$ if $e_h \in \mathcal{C}$ and $g(e_h) = 0$ if $e_h \notin \mathcal{C}$.
The matrix $M_{\mathcal{C}}^{\mathcal{B}}(g \circ A \circ f)$ is exactly B, so that $g \circ A \circ f$ is an isomorphism. In particular, $\dim \operatorname{Im}(A) \geq \dim \operatorname{Im}(g \circ A \circ f) = k$.

(b) Let $C_{i_1}, \dots, C_{i_k}$ be linearly independent columns of A and let $R_{j_1}, \dots, R_{j_k}$ be the linearly indepedent rows of A.
If $\{e_1, \dots, e_n\}$ is the standand basis of $\mathbb{F}^n$, let $\{e_1^*, \dots, e_n^*\}$ be the standard dual basis of $(\mathbb{F}^n)^*$.
Remember that $A^t : (\mathbb{F}^n)^* \longrightarrow (\mathbb{F}^n)^*$. Thus, every row of A corresponds to a functional on $\mathbb{F}^n$: the row R_j corresponds to the functional $A^t(e_j^*)$.
Remember also that $\ker(A) = \operatorname{Ann}(\operatorname{Im}(A^t))$.
As the rows $R_{j_1}, \dots, R_{j_k}$ are linearly independent, the set $\{A^t(e_{j_1}), \dots, A^t(e_{j_k})\}$ is a basis of $\operatorname{Im}(A^t)$.
Now we show that the columns of B are linearly independent.
Let $\alpha_1, \dots, \alpha_k \in \mathbb{F}$ such that $\alpha_1 B_{r1} + a_2 B_{r2} + \cdots + a_k B_{rk} = 0$ for every $r = 1, \dots, k$. This is the same as saying that the vector $\alpha_1 C_{i_1} + \cdots + \alpha_k C_{i_k} = A(\alpha_1 e_{i_1} + \cdots + \alpha_k e_{i_k})$ has the j_h-th entry equal to zero for $h = 1, \dots, k$. Thus, the vector $\alpha_1 e_{i_1} + \cdots + \alpha_k e_{i_k}$ is annihilated by $\{A^t(e_{j_1}), \dots, A^t(e_{j_k})\}$ and so by $\operatorname{Im}(A^t)$. Hence, the vector $\alpha_1 e_{i_1} + \cdots + \alpha_k e_{i_k}$ belongs to $\ker(A)$ and so $\alpha_1 C_{i_1} + \cdots + \alpha_k C_{i_k} = 0$.
As the columns $C_{i_1}, \dots, C_{i_k}$ are linearly independent, we obtain $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$.

Problem 4.

(a) In writing $\det(M) = \displaystyle\sum_{\sigma \in \mathfrak{S}_{n+m}} \varepsilon(\sigma) M_{\sigma(1),1} \cdots M_{\sigma(n+m),n+m}$ we discover that the only permutations that give a nonzero monomial are those $\sigma : \{1, 2, \dots, n+m\} \to \{1, 2, \dots, n+m\}$ that send the subset $\{1, 2, \dots, n\}$ to itself (and consequently the subset $\{n+1, \dots, n+m\}$ to itself). Call $\mathfrak{S}_{n,m} \subset \mathfrak{S}_{n+m}$ the subset of such permutations.
Consider the bijection $\mathfrak{S}_n \times \mathfrak{S}_m \longrightarrow \mathfrak{S}_{n,m}$ that sends (α, β) to the permutation σ defined

as $\sigma(i) = \alpha(i)$ if $1 \le i \le n$ and $\sigma(i) = n + \beta(i - n)$ if $n + 1 \le i \le n + m$. Notice that $\varepsilon(\sigma) = \varepsilon(\alpha)\varepsilon(\beta)$.
Then we can clearly rewrite $\det(M)$ as $\displaystyle\sum_{\sigma \in \mathfrak{S}_{n,m}} \varepsilon(\sigma) A_{\sigma(1),1} \cdots A_{\sigma(n),n} B_{\sigma(n+1),1} \cdots B_{\sigma(n+m),m} =$ $\displaystyle\left(\sum_{\alpha \in \mathfrak{S}_n} \varepsilon(\alpha) A_{\alpha(1),1} \cdots A_{\alpha(n),n}\right)\left(\sum_{\beta \in \mathfrak{S}_m} \varepsilon(\beta) B_{\beta(1),1} \cdots B_{\beta(m),m}\right) = \det(A)\det(B)$.
The same argument, applied to $A - \lambda I_n$, $B - \lambda I_m$ and $M - \lambda I_{n+m}$, shows that $p_M(\lambda) = \det(M - \lambda I) = \det(A - \lambda I)\det(B - \lambda I) = p_A(\lambda)p_B(\lambda)$.

(b) If A and B are both diagonalizable, then there exist invertible matrices $P \in \mathcal{M}_{n\times n}(\mathbb{F})$ and $Q \in \mathcal{M}_{m\times m}(\mathbb{F})$ such that PAP^{-1} and QBQ^{-1} are diagonal. Then, the matrix $R = \left(\begin{array}{c|c} P & 0 \\ \hline 0 & Q \end{array}\right)$ has the property that RMR^{-1} is diagonal.
Vice versa, assume M diagonalizable and call $E_e \subseteq \mathbb{F}^{n+m}$ the eigenspace of M corresponding to the eigenvalue e. Define subspaces $W_1 = \operatorname{span}\{e_1, \ldots, e_n\}$ and $W_2 = \operatorname{span}\{e_{n+1}, \ldots, e_{n+m}\}$ of $\mathbb{F}^{n+m}$. Then $\mathbb{F}^{n+m} = W_1 \oplus W_2$ and $A(W_1) \subseteq W_1$ and $B(W_2) \subseteq W_2$.
Notice that the matrix A is diagonalizable if and only if the homomorphism $\tilde{A} := M\Big|_{W_1}^{W_1} : W_1 \longrightarrow W_1$ is. This happens if and only if W_1 is the direct sum of the eigenspaces of $\tilde{A}$ (similar statement for B, $\tilde{B}$ and W_2).
We claim that $E_e = E_{e,1} \oplus E_{e,2}$, where $E_{e,i} = E_e \cap W_i$ for $i = 1, 2$.
Clearly, $E_{e,1} \cap E_{e,2} \subseteq W_1 \cap W_2 = \{0\}$.
If $v \in E_e$, then $v = w_1 + w_2$ with $w_1 \in W_1$ and $w_2 \in W_2$. Moreover, $ew_1 + ew_2 = ev = Mv = Mw_1 + Mw_2 = \tilde{A}w_1 + \tilde{B}w_2$, so that $\tilde{A}w_1 = ew_1$ and $\tilde{B}w_2 = ew_2$, which shows that $w_i \in E_{e,i}$ and $E_{e,1} + E_{e,2} = E_e$.
If $e_1, \ldots, e_k$ are the eigenvalues of M, then

$$\mathbb{F}^{n+m} = E_{e_1} \oplus \cdots \oplus E_{e_k} = (E_{e_1,1} \oplus \cdots \oplus E_{e_k,1}) \oplus (E_{e_1,2} \oplus \cdots \oplus E_{e_k,2}) = W_1 \oplus W_2$$

so that $W_1 = E_{e_1,1} \oplus \cdots \oplus E_{e_k,1}$ and $W_2 = E_{e_1,2} \oplus \cdots \oplus E_{e_k,2}$. As $E_{e_h,1} \subseteq W_1$ is an eigenspace for $\tilde{A}$ and $E_{e_h,2} \subseteq W_2$ is an eigenspace fo $\tilde{B}$, this shows that $\tilde{A}$ and $\tilde{B}$ (and so A and B) are diagonalizable.

(c) Notice that $N - \lambda I = QR$, where $Q = \left(\begin{array}{c|c} I_{n\times n} & C_{n\times m} \\ \hline 0_{m\times n} & B_{m\times m} - \lambda I_{m\times m} \end{array}\right)$ and $R = \left(\begin{array}{c|c} A_{n\times n} - \lambda I_{\times n} & 0_{n\times m} \\ \hline 0_{m\times n} & I_{m\times m} \end{array}\right)$, so that $p_N(\lambda) = \det(Q)\det(R)$.
Moreover, $\det(Q)$ is equal to $\det(Q')$ where $Q' = \left(\begin{array}{c|c} I_{n\times n} & 0_{n\times m} \\ \hline 0_{m\times n} & B_{m\times m} \end{array}\right)$, because Q' is obtained from Q by elementary operations on the columns.
From part (a), we have $\det(Q) = p_B(\lambda)$ and $\det(R) = p_A(\lambda)$. Hence, $p_N(\lambda) = p_A(\lambda)p_B(\lambda)$.

(d) N can be diagonalizable even if C is not zero.
As an example, consider $N = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{F})$.
Clearly, N is diagonalizable, because $\{e_1, e_1 - e_2\}$ is a basis of $\mathbb{F}^2$ of eigenvectors for N.