18.700 - Fall 2006 - Problem Set 5 (55 points)

(Due on **Tuesday, Oct 31st**)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". All solutions must be supported by proofs or counterexamples. NO LATE HOMEWORK IS ALLOWED.

Problem 1. (10 points: 3+3+4)

Let $L : \mathbb{C}^4 \longrightarrow \mathbb{C}^4$ be the homomorphism defined by

$$L = \begin{pmatrix} 1 & 0 & -3 & 1 \\ 2 & 1 & 1 & -1 \\ -1 & 2 & c-1 & 0 \\ 0 & c & 2 & 1 \end{pmatrix}$$

where c is an indeterminate.

(a) Compute $\det(A)$.

(b) For which values of c is A not invertible? In these cases, find a basis for $\ker(A)$.

(c) Compute the inverse of A for $c = 1$.

Problem 2. (18 points: 2+3+4+4+5)

Let $A, B : \mathbb{F}^4 \longrightarrow \mathbb{F}^4$ be 4×4 matrices with coefficients in $\mathbb{F}$.
Define an equivalence relation $\sim$ declaring $A \sim B$ if and only if there exists an invertible 4×4 matrix $M \in \mathcal{M}_{4\times 4}(\mathbb{F})$ such that $A = BM$.

(a) Verify that $\sim$ is an equivalence relation.

(b) Is the trace invariant under this equivalence relation?
Is the determinant invariant under this equivalence relation?

(c) Is the kernel an invariant? Is the image an invariant?

(c) Find a "canonical form" in each equivalence class.

(d) Show that there is a bijection between the equivalence classes and the vector subspaces of $\mathbb{F}^4$.

Problem 3. (13 points: 3+6+4)

Consider the following $n \times n$ matrix (called *Vandermonde matrix*)

$$V(x_1, \dots, x_n) := \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix}$$

that is, $V(x_1, \dots, x_n)_{ij} = x_i^{j-1}$. This is a matrix with entries in the set $\mathbb{F}[x_1, \dots, x_n]$ of polynomials in the variables $x_1, \dots, x_n$ with coefficients in $\mathbb{F}$.
Also $\det(V(x_1, \dots, x_n))$ will be a polynomial in $\mathbb{F}[x_1, \dots, x_n]$.

(a) Prove that $\det(V(x_1, \dots, x_n))$ is a homogeneous[1] polynomial in $\mathbb{F}[x_1, \dots, x_n]$ of total degree $(n-1)n/2$.

(b) Show that, for every $1 \leq i < j \leq n$, the polynomial $\det(V(x_1, \dots, x_n))$ is a multiple of $(x_j - x_i)$. Conclude that $\det(V(x_1, \dots, x_n)) = \alpha \prod_{1 \leq i < j \leq n} (x_j - x_i)$ for some $\alpha \in \mathbb{F}$.

(c) Prove that $\alpha = 1$, so that $\det(V(x_1, \dots, x_n)) = \prod_{1 \leq i < j \leq n} (x_j - x_i)$.

Problem 4. (14 points)

Consider the subset $\mathcal{S}$ of $\mathcal{M}_{n \times n}(\mathbb{F})^*$ consisting of all linear functionals $\varphi : \mathcal{M}_{n \times n}(\mathbb{F}) \longrightarrow \mathbb{F}$ such that $\varphi(AB) = \varphi(BA)$ for every $A, B \in \mathcal{M}_{n \times n}(\mathbb{F})$.
Show that $\mathcal{S}$ is a vector subspace of $\mathcal{M}_{n \times n}(\mathbb{F})^*$ of dimension 1 and all functionals in $\mathcal{S}$ are multiples of the trace $\mathrm{tr} : \mathcal{M}_{n \times n}(\mathbb{F}) \longrightarrow \mathbb{F}$.
(Hint: pick a simple basis $\mathcal{B}$ of $\mathcal{M}_{n \times n}(\mathbb{F})$ and see what the condition $\varphi(AB) = \varphi(BA)$ implies when $A, B \in \mathcal{B}$.)

[1] The **total degree** of a nonzero monomial in $\mathbb{F}[x_1, \dots, x_n]$ is the sum of the exponents of the x_i's appearing in the monomial. For example, the total degree of $2x_1^2 x_2^3 x_4^6$ is 11. The total degree of a nonzero polynomial in $\mathbb{F}[x_1, \dots, x_n]$ is the maximum of the total degrees of its monomials. A polynomial is **homogeneous** if all its monomials have the same total degree.