18.700 - Fall 2006 - Solutions of Problem Set 5

Problem 1.

(a) By direct computation, $\det(L) = 3c^2 - 4c - 32 = (3c+8)(c-4)$.

(b) For $c = 4$ and $c = -8/3$.

For $c = 4$, $L = \begin{pmatrix} 1 & 0 & -3 & 1 \\ 2 & 1 & 1 & -1 \\ -1 & 2 & 3 & 0 \\ 0 & 4 & 2 & 1 \end{pmatrix}$. The first, second and fourth columns are clearly linearly independent, so $\ker(L)$ has dimension 1, with basis $\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 2 \end{pmatrix} \right\}$.

For $c = -8/3$, $L = \begin{pmatrix} 1 & 0 & -3 & 1 \\ 2 & 1 & 1 & -1 \\ -1 & 2 & 5/3 & 0 \\ 0 & 8/3 & 2 & 1 \end{pmatrix}$. The first, second and fourth column are again linearly independent, so $\ker(L)$ has again dimension 1, with basis $\left\{ \begin{pmatrix} 1 \\ 39 \\ 21 \\ 62 \end{pmatrix} \right\}$.

(c) For $c = 1$, we have

$$L = \begin{pmatrix} 1 & 0 & -3 & 1 \\ 2 & 1 & 1 & -1 \\ -1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

and $\det(L) \neq 0$, so that L in invertible.

By direct computation, $L^{-1} = \dfrac{1}{33} \begin{pmatrix} 6 & 10 & -7 & 4 \\ 3 & 5 & 13 & 2 \\ -6 & 1 & -4 & 7 \\ 9 & -7 & -5 & 17 \end{pmatrix}$.

Problem 2. Let $A, B : \mathbb{F}^4 \longrightarrow \mathbb{F}^4$ be 4×4 matrices with coefficients in $\mathbb{F}$.
Define an equivalence relation $\sim$ declaring $A \sim B$ if and only if there exists an invertible 4×4 matrix $M \in \mathcal{M}_{4\times4}(\mathbb{F})$ such that $A = BM$.

(a) Reflexivity: $A \sim A$, because $A = AI$.
Commutativity: if $A \sim B$, then $A = BM$, so $B = AM^{-1} \implies B \sim A$.
Transitivity: if $A \sim B$ and $B \sim C$, then $A = BM$ and $B = CN$, then $A = C(NM)$ and NM is invertible, so $A \sim C$.

(b) Notice that, if A is invertible, then $A \sim I$. So invertible matrices are equivalent.

In general, for a field $\mathbb{F}$ with more than two elements (i.e. any field except $\mathbb{Z}/2$), take $a \in \mathbb{F}$ different from 0 and 1 and let A be the diagonal matrix with $A_{11} = a$ and $A_{jj} = 1$ for $j \neq 1$. Then $\det(A) = a \neq 0$, so A is invertible and so $A \sim I$, but $\det(A) = a \neq 1 = \det(I)$, so det is not invariant.
Moreover, $\operatorname{tr}(A) = (n-1) + a \neq n = \operatorname{tr}(I)$, so the trace is not invariant either. [If $\mathbb{F} = \mathbb{Z}/2$, then $\det(A) = \bar{1}$ if A is invertible and $\det(A) = \bar{0}$ if A is not invertible. So, in this very particular case, det is invariant.]

(c) The kernel is not invariant. In fact, if A is the diagonal matrix with $A_{11} = 1$ and $A_{jj} = 0$ for $j \neq 1$, then $\ker(A) = \operatorname{span}\{e_2, e_3, e_4\}$. If $M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, then $\ker(AM) = \operatorname{span}\{e_1, e_2, e_3\} \neq \ker(A)$.
The image is invariant. In fact, if $A = BM$, then $\operatorname{Im}(A) = \operatorname{Im}(BM) \subseteq \operatorname{Im}(B)$. But $B = AM^{-1}$, so that $\operatorname{Im}(B) = \operatorname{Im}(AM^{-1}) \subseteq \operatorname{Im}(A)$ and so $\operatorname{Im}(A) = \operatorname{Im}(B)$.
Equivalently, right-multiplication by invertible matrices corresponds to operating on the columns by a sequence of elementary operations. These operations preserve the image.

(d) Multiplying by invertible matrices on the right corresponds to operating on the columns by a sequence of elementary operations. Thus, given a matrix A, we can find an invertible M such that $B = AM$ is in reduced column-Echelon form. Thus, in each equivalence class, we can find a matrix in reduced column-Echelon form. We declare matrices in reduced column-Echelon form to be our canonical forms. We still have to prove that in each equivalence class we can find *only one* matrix in reduced column-Echelon form. See Theorem 0.5.5, at page 54-55 of Jacob's book, but applied to the columns.

(e) Call $[A]$ the equivalence class of matrices that contains A. Define

$$\begin{tikzcd} \Phi : \mathcal{M}_{n\times n}(\mathbb{F})/\sim \arrow[r] & \{\text{vector subsspaces of } \mathbb{F}^4\} \\ {[A]} \arrow[r, mapsto] & \operatorname{Im}(A) \end{tikzcd}$$

This is well-defined, because the image is invariant.
Clearly, this map is surjective because for every subspace $W \subseteq \mathbb{F}^4$, I can find a complement $Z \subseteq \mathbb{F}^4$ so that $W \oplus Z = \mathbb{F}^4$ and I can define $A : \mathbb{F}^4 \longrightarrow \mathbb{F}^4$ as $A(w+z) = w$ for every $w \in W$ and $z \in Z$. Then $\operatorname{Im}(A) = W$ and so $\Phi([A]) = W$.
For the injectivity, we have to show that: if $\operatorname{Im}(A) = \operatorname{Im}(B)$, then $A \sim B$.
Let $\{w_1, \ldots, w_k\}$ be a basis of $\operatorname{Im}(A) = \operatorname{Im}(B)$. Let $\{z_1, \ldots, z_{4-k}\}$ be a basis of $\ker(A)$ and $\{z'_1, \ldots, z'_{4-k}\}$ be a basis of $\ker(B)$.
Moreover, let $v_i \in \mathbb{F}^4$ such that $A(v_i) = w_i$ and let $v'_i \in \mathbb{F}^4$ such that $B(v'_i) = w_i$ for $i = 1, \ldots, k$. Then $\mathcal{B} = \{v_1, \ldots, v_k, z_1, \ldots, z_{4-k}\}$ and $\mathcal{B}' = \{v'_1, \ldots, v'_k, z'_1, \ldots, z'_{4-k}\}$ are both bases of $\mathbb{F}^4$. Define $M : \mathbb{F}^4 \longrightarrow \mathbb{F}^4$ as the isomorphism such that $M(v'_i) = v_i$ for $i = 1, \ldots, k$ and $M(z'_j) = z_j$ for $j = 1, \ldots, 4-k$. Then, $B = AM$ because B and AM agree on $\mathcal{B}'$ by construction. Thus $A \sim B$.

Problem 3.

(a) Each monomial in $\det(V(x_1,\ldots,x_n))$ has one entry for each column, so that it has total degree $0+1+2+\cdots+(n-2)+(n-1)=n(n-1)/2$.

(b) If we set $x_j = x_i$, then V has two columns which are equal, that it is not invertible and its determinant is zero. Thus, the polynomial $\det(V(x_1,\ldots,x_n))$ is divisible by $(x_j - x_i)$.
Hence, $\det(V(x_1,\ldots,x_n))$ is divisible by $\displaystyle\prod_{1\le i<j\le n}(x_j - x_i)$. As both have total degree $n(n-1)/2$, then $\det(V(x_1,\ldots,x_n)) = \alpha \displaystyle\prod_{1\le i<j\le n}(x_j - x_i)$, with α of degree zero, so $\alpha \in \mathbb{F}$.

(c) Using the definition of det, the only monomial in $\det(V(x_1,\ldots,x_n))$ proportional to $x_2 x_3^2 x_4^3 \cdots x_n^{n-1}$ is obtained from $\sigma = id$. Thus it occurs with coefficient $\varepsilon(id) = 1$.
In $\prod_{1\le i<j\le n}(x_j - x_i)$, the same monomial occurs with coefficient 1, so that $\alpha = 1$.

Problem 4.

Call $E^{(ij)}$ the $n \times n$ matrix with entry 1 at position (i,j) and zero elsewhere.
Clearly, $E^{(ij)}E^{(hk)} = 0$ if $j \neq h$ and $E^{(ij)}E^{(jk)} = E^{(ik)}$.
For $i \neq k$, we have $\varphi(E^{(jk)}E^{(ij)}) = \varphi(0) = 0$. On the other hand, $\varphi(E^{(jk)}E^{(ij)}) = \varphi(E^{(ij)}E^{(jk)}) = \varphi(E^{(ik)})$, so that $\varphi(E^{(ik)}) = 0$ if $i \neq k$.
Instead, $\varphi(E^{(ii)}) = \varphi(E^{(ij)}E^{(ji)}) = \varphi(E^{(ji)}E^{(ij)}) = \varphi(E^{(jj)})$.
Call $\alpha = \varphi(E^{(11)})$.
Thus, we have found that $\varphi(E^{(ij)}) = \alpha \cdot \mathrm{tr}(E^{(ij)})$.
As $\{E^{(ij)}\}$ is a basis of $\mathcal{M}_{n\times n}(\mathbb{F})$, this implies that $\varphi = \alpha \cdot \mathrm{tr}$.
As $\mathrm{tr} \neq 0$, then it generates a subspace of $\mathcal{M}_{n\times n}(\mathbb{F})^*$ of dimension 1.