18.700 - Fall 2006 - Problem Set 4 (48 points)

(Due on **Tuesday, Oct 24th**)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". All solutions must be supported by proofs or counterexamples. NO LATE HOMEWORK IS ALLOWED.

Problem 1. (12 points: 3+3+2+4)

Let $L : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be the homomorphism defined by

$$L = \begin{pmatrix} 1 & -1 & -2 & 1 \\ 0 & -2 & -3 & 1 \\ -2 & -2 & -2 & 0 \\ -1 & 1 & 2 & -1 \end{pmatrix}$$

(a) What is the reduced row-Echelon form of L?

(b) Determine a basis of $\ker(L)$.

(c) Determine a basis of $\mathrm{Im}(L)$.

(d) Find the inverse of $(L + I)$ (where I is the 4×4 identity matrix).

Problem 2. (12 points: 3+3+3+3)

Show that the following subsets of $\mathcal{M}_{n\times n}(\mathbb{F})$ are vector subspaces and find their dimension.

(a) The subset of upper triangular matrices; the subset of strictly upper triangular matrices.

(b) The subset of matrices A such that $\mathrm{tr}(A) = 0$. (Remember that $\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}$.)

(c) The subset of *symmetric matrices*, i.e. matrices A such that $A^t = A$.

(d) The subset of *skew-symmetric matrices*, i.e. matrices A such that $A^t = -A$.

Problem 3. (14 points: 2+3+4+5)

Let V be a vector space over $\mathbb{F}$ and let $G : V \longrightarrow V$ be a homomorphism.

Define $\mathrm{ev}_G : \mathbb{F}[t] \longrightarrow \hom_{\mathbb{F}}(V, V)$ as $\mathrm{ev}_G(p(t)) = p(G)$.
(Example: if $p(t) = 5t^2 - 8t + 3$, then $p(G) = 5G^2 - 8G + 3I$, where I is the identity homomorphism from V to V.)

(a) Show that ev_G is a homomorphism.

(b) Show that, if $H : V \longrightarrow V$ is an isomorphism, then $\mathrm{ev}_{HGH^{-1}}(p(t)) = H \cdot \mathrm{ev}_G(p(t)) \cdot H^{-1}$. Conclude that $\ker(\mathrm{ev}_{HGH^{-1}}) = \ker(\mathrm{ev}_G)$.

(c) Compute $\ker(\mathrm{ev}_G)$ in the case of $V = \mathbb{F}^n$ and $G : \mathbb{F}^n \longrightarrow \mathbb{F}^n$ a diagonal matrix.

(d) Let $V = \mathbb{F}^n$ and show that, for every $k = 1, \ldots, n$, there exists a $G : \mathbb{F}^n \longrightarrow \mathbb{F}^n$ such that $\ker(\mathrm{ev}_G) = \{\text{polynomials which are multiples of } t^k\}$.

Problem 4. (10 points)

Let V be a vector space over $\mathbb{F}$ and let $f_1, \ldots, f_k, g$ be elements of V^*. Prove that

$$\exists a_1, \ldots, a_k \in \mathbb{F} \text{ such that } g = \sum_{i=1}^{k} a_i f_i \iff \bigcap_{i=1}^{k} \ker(f_i) \subseteq \ker(g)$$