18.700 - Fall 2006 - Problem Set 4 (48 points)

(Due on **Tuesday, Oct 24th**)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". All solutions must be supported by proofs or counterexamples. NO LATE HOMEWORK IS ALLOWED.

Problem 1.

(a) We start with $L = \begin{pmatrix} 1 & -1 & -2 & 1 \\ 0 & -2 & -3 & 1 \\ -2 & -2 & -2 & 0 \\ -1 & 1 & 2 & -1 \end{pmatrix}$. Adding the first row to the last row and twice the first row to the third row, we obtain $\begin{pmatrix} 1 & -1 & -2 & 1 \\ 0 & -2 & -3 & 1 \\ 0 & -4 & -6 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Multiplying the second row by $-\frac{1}{2}$ and then summing four times the second row to the third row, we obtain $\begin{pmatrix} 1 & -1 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Finally, adding the second row to the first row, we obtain the reduced row-Echelon form of L, that is $\begin{pmatrix} 1 & 0 & -1/2 & 1/2 \\ 0 & 1 & 3/2 & -1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

(b) The kernel of L is the same as the kernel of its reduced row-Echelon form (because elementary operations on the rows preserve the kernel). So, $\begin{pmatrix} 1 & 0 & -1/2 & 1/2 \\ 0 & 1 & 3/2 & -1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a - c/2 + d/2 \\ b + 3c/2 - d/2 \\ 0 \\ 0 \end{pmatrix}$, from which we get that $\left\{ \begin{pmatrix} 1 \\ -3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 2 \end{pmatrix} \right\}$ is a basis of $\ker(L)$.

(c) To deterimine a basis of $\mathrm{Im}(L)$ we could use that the image is preserved under elementary operations on the columns. However, we already know that $\dim(\mathrm{Im}(L)) = 4 - \dim(\ker(L)) = 2$ and that the columns of L span the image of L. As the first two columns are a set of linearly independent vectors, then a basis of $\mathrm{Im}(L)$ is $\left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ -2 \\ 1 \end{pmatrix} \right\}$.

(d) We want to start with $(L + I|I)$ and reduce $L + I$ in reduced row-Echelon form (which will be I_4, if $L + I$ is invertible), operating on the rows made of 8 entries. On the right side, we will obtain the inverse of $L+I$.

Start with $(L+I|I)=\left(\begin{array}{cccc|cccc} 2 & -1 & -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -3 & 1 & 0 & 1 & 0 & 0 \\ -2 & -2 & -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$. Multiplying the first row by 1/2, we get $\left(\begin{array}{cccc|cccc} 1 & -1/2 & -1 & 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & -1 & -3 & 1 & 0 & 1 & 0 & 0 \\ -2 & -2 & -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$. Adding the first row to the fourth row and twice the first row to the second row, we obtain $\left(\begin{array}{cccc|cccc} 1 & -1/2 & -1 & 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & -1 & -3 & 1 & 0 & 1 & 0 & 0 \\ 0 & -3 & -3 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1/2 & 1 & 1/2 & 1/2 & 0 & 0 & 1 \end{array}\right)$. Multiply the second row by -1 and then adding 3 times the second row to the third row and $-1/2$ times the second row to the last row, we obtain $\left(\begin{array}{cccc|cccc} 1 & -1/2 & -1 & 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 6 & -2 & 1 & -3 & 1 & 0 \\ 0 & 0 & -1/2 & 1 & 1/2 & 1/2 & 0 & 1 \end{array}\right)$. Multiply the third row by 1/6 and then add 1/2 times the third row to the fourth row, getting $\left(\begin{array}{cccc|cccc} 1 & -1/2 & -1 & 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1/3 & 1/6 & -1/2 & 1/6 & 0 \\ 0 & 0 & 0 & 5/6 & 7/12 & 1/4 & 1/12 & 1 \end{array}\right)$. Multiply the last row by 6/5 and get $\left(\begin{array}{cccc|cccc} 1 & -1/2 & -1 & 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1/3 & 1/6 & -1/2 & 1/6 & 0 \\ 0 & 0 & 0 & 1 & 7/10 & 3/10 & 1/10 & 6/5 \end{array}\right)$. Summing 1/3 times the fourth row to the third row, the fourth row to the second row and $-1/2$ times the fourth row to the first row, we obtain $\left(\begin{array}{cccc|cccc} 1 & -1/2 & -1 & 0 & 3/20 & -3/20 & -1/20 & -3/5 \\ 0 & 1 & 3 & 0 & 7/10 & -7/10 & 1/10 & 6/5 \\ 0 & 0 & 1 & 0 & 2/5 & -2/5 & 1/5 & 2/5 \\ 0 & 0 & 0 & 1 & 7/10 & 3/10 & 1/10 & 6/5 \end{array}\right)$. Adding the third row to the first row and -3 times the third row to the second row, we get $\left(\begin{array}{cccc|cccc} 1 & -1/2 & 0 & 0 & 11/20 & -11/20 & 3/20 & -1/5 \\ 0 & 1 & 0 & 0 & -1/2 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 2/5 & -2/5 & 1/5 & 2/5 \\ 0 & 0 & 0 & 1 & 7/10 & 3/10 & 1/10 & 6/5 \end{array}\right)$. Adding 1/2 times the second row to the first row, we finally have $\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3/10 & -3/10 & -1/10 & -1/5 \\ 0 & 1 & 0 & 0 & -1/2 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 2/5 & -2/5 & 1/5 & 2/5 \\ 0 & 0 & 0 & 1 & 7/10 & 3/10 & 1/10 & 6/5 \end{array}\right)$, so that

$$(L+I)^{-1}=\frac{1}{10}\begin{pmatrix} 3 & -3 & -1 & -2 \\ -5 & 5 & -5 & 0 \\ 4 & -4 & 2 & 4 \\ 7 & 3 & 1 & 12 \end{pmatrix}.$$

Problem 2.

(a) Call $E_{ij} \in \mathcal{M}_{n\times n}(\mathbb{F})$ the matrix that has 1 at the entry (i,j) and 0 elsewhere.

Clearly, if both A, B are upper triangular $n \times n$ matrices, then $A_{ij} = 0$ and $B_{ij} = 0$ whenever $i > j$. As a consequence $(A+B)_{ij} = A_{ij} + B_{ij} = 0$ whenever $i > j$, so that $A + B$ is upper triangular. If $\lambda \in \mathbb{F}$ and A is upper triangular (resp. strictly upper triangular), then $(\lambda A)_{ij} = 0$ whenever $i > j$, so that λA is upper triangular. Hence, the subset of upper triangular matrices in $\mathcal{M}_{n\times n}(\mathbb{F})$ is a vector subspace.
The argument for strictly upper triangular matrices is analogous.

A basis for the subset of $\mathcal{M}_{n\times n}(\mathbb{F})$ of upper triangular matrices is given by $\{E_{ij} \,|\, 1 \leq i \leq j \leq n\}$, so that the subspace of upper triangular matrices has dimension $n(n+1)/2$.

A basis for the subset of $\mathcal{M}_{n\times n}(\mathbb{F})$ of strictly upper triangular matrices is given by $\{E_{ij} \,|\, 1 \leq i < j \leq n\}$, so that the subspace of strictly upper triangular matrices has dimension $n(n-1)/2$.

(b) $\mathrm{tr} : \mathcal{M}_{n\times n}(\mathbb{F}) \longrightarrow \mathbb{F}$ is a homomorphism, so that the subspace of matrices with zero trace is $\ker(\mathrm{tr})$ and so it is a vector subspace of $\mathcal{M}_{n\times n}(\mathbb{F})$ of dimension $n^2 - 1$ (because $\mathrm{Im}(\mathrm{tr})$ has dimension 1, as $\mathrm{tr}(E_{1,1}) = 1 \neq 0$).

A basis for this vector subspace is given by $\mathcal{B} = \{E_{ij} \,|\, i \neq j\} \cup \{E_{jj} - E_{11} \,|\, 2 \leq j \leq n\}$.
Notice that all matrices in $\mathcal{B}$ have zero trace, they are linearly independent and they are exactly $n^2 - 1$. Hence, they are a basis for the subspace.

(c) If A, B are symmetric, then $(A+B)^t = A^t + B^t = A + B$, so $A + B$ is symmetric.
If A is symmetric and $\lambda \in \mathbb{F}$, then $(\lambda A)^t = \lambda A^t = \lambda A$, so that λA is symmetric. Hence, the subset of symmetric matrices is a vector subspace of $\mathcal{M}_{n\times n}(\mathbb{F})$.

A basis for this subspace is given by $\{E_{ij} + E_{ji} \,|\, 1 \leq i < j \leq n\} \cup \{E_{ii} \,|\, 1 \leq i \leq n\}$, so that the dimension of the subspace of symmetric matrices is $n(n+1)/2$.

(d) If A, B are skew-symmetric, then $(A+B)^t = A^t + B^t = -A - B = -(A+B)$, so $A + B$ is skew-symmetric.
If A is skew-symmetric and $\lambda \in \mathbb{F}$, then $(\lambda A)^t = \lambda A^t = \lambda(-A) = -(\lambda A)$, so that λA is skew-symmetric. Hence, the subset of skew-symmetric matrices is a vector subspace of $\mathcal{M}_{n\times n}(\mathbb{F})$.

A basis for this subspace is given by $\{E_{ij} - E_{ji} \,|\, 1 \leq i < j \leq n\}$, so that the dimension of the subspace of skew-symmetric matrices is $n(n-1)/2$.

Problem 3.

(a) Given $p, q \in \mathbb{F}[t]$, by definition $\mathrm{ev}_G(p+q) = (p+q)(G) = p(G) + q(G) = \mathrm{ev}_G(p) + \mathrm{ev}_G(q)$.
Given $p \in \mathbb{F}[t]$ and $\lambda \in \mathbb{F}$, we have $\mathrm{ev}_G(\lambda p) = (\lambda p)(G) = \lambda p(G) = \lambda \mathrm{ev}_G(p)$.

(b) As ev_G and $\mathrm{ev}_{HGH^{-1}}$ are both homomorphisms and $\{t^k \in \mathbb{F}[t] \,|\, k \in \mathbb{N}\}$ is a set of generators (and even a basis) of $\mathbb{F}[t]$, it is sufficient to check that $\mathrm{ev}_G(t^k) = \mathrm{ev}_{HGH^{-1}}(t^k)$ for every $k \in \mathbb{N}$.
It is very simple, but let's prove it formally by induction on $k \geq 0$.
For $k = 0$, $H\mathrm{ev}_G(1)H^{-1} = HIH^{-1} = I = \mathrm{ev}_{HGH^{-1}}(1)$.
Assume now it proven for k: we want to show it for $k+1$.
By definition, $\mathrm{ev}_{HGH^{-1}}(t^{k+1}) = (HGH^{-1})^{k+1} = (HGH^{-1})^k(HGH^{-1}) = HG^kH^{-1}HGH^{-1} = HG^{k+1}H^{-1} = H\mathrm{ev}_G(t^{k+1})H^{-1}$.
Then, as H is invertible, $\mathrm{ev}_G(p) = 0$ if and only if $\mathrm{ev}_{HGH^{-1}}(p) = 0$; and so $\ker(\mathrm{ev}_G) = \ker(\mathrm{ev}_{HGH^{-1}})$.

(c) Call $\mathrm{diag}(a_1, \ldots, a_n)$ a square $n \times n$ matrix with entry a_i at position (i,i) for $i - 1, \ldots, n$ and zeroes elsewhere.

Let $G = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, so that $\mathrm{ev}_G(p) = \mathrm{diag}(p(\lambda_1), \ldots, p(\lambda_n))$. Clearly, $\mathrm{ev}_G(p) = 0$ if and only if p is vanishes on $\lambda_1, \ldots, \lambda_n$. Let $\{r_1, \ldots, r_k\} = \{\lambda_1, \ldots, \lambda_n\}$ for suitable distinct $r_1, \ldots, r_k$ (necessarily, $k \leq n$). Call $m(t) = \prod_{i=1}^{k}(t - r_i)$.

The kernel of ev_G is $\{q(t) \in \mathbb{F}[t] \,|\, q(t) \text{ is a multiple of } m(t)\}$.

(d) Call $N_{n,n}$ the $n \times n$ matrix that has entries 1 at position (i,j) for $j - i = 1$. A direct computation shows that $(N_{n,n})^a$ is the matrix that has entries 1 at position (i,j) for $j - i = a$ and zeroes elsewhere.

Clearly, $N_{n,n}^n = 0$ and $\{N_{n,n}, (N_{n,n})^2, \ldots, (N_{n,n})^{n-1}\}$ is a set of linearly independent vectors in $\mathcal{M}_{n\times n}(\mathbb{F})$.

Let $p(t) = \sum_{i=0}^{d} a_i t^i \in \mathbb{F}[t]$. Then $\mathrm{ev}_{N_1}(p) = \sum_{i=0}^{d} a_i (N_{n,n})^i$, which is zero if and only if $a_j = 0$ for all $0 \leq j \leq n-1$, that is if and only if $p(t)$ is a multiple of t^n. Thus, for $k = n$, we can take $G = N_{n,n}$.

For $k = 1$, we can clearly take $G = 0$.

For k between 2 and $n-1$, we define

$$N_{k,n} = \left(\begin{array}{c|c} N_{k,k} & 0 \\ \hline 0 & 0_{n-k} \end{array}\right)$$

that is, $(N_{k,n})_{ij}$ is 1 if $j - i = 1$ and $2 \leq j \leq k$, and zero otherwise.

Similarly to what done before, $(N_{k,n})^a$ has entries 1 if $j - i = a$ and $2 \leq j \leq k$, and zeroes elsewhere. Hence, $(N_{k,n})^a = 0$ for $a \geq k$ and $\{N_{k,n}, (N_{k,n})^2, \ldots, (N_{k,n})^{k-1}\}$ is a set of linearly independent vectors in $\mathcal{M}_{n\times n}(\mathbb{F})$.

The same argument as before shows that $\ker(\mathrm{ev}_{N_{k,n}})$ consists of polynomials in $\mathbb{F}[t]$ that are multiples of t^k.

Problem 4.

Proof of $\boxed{\Longrightarrow}$
Let $v \in V$ that belongs to $\ker(f_i)$ for $i = 1, \dots, k$. Then $f_i(v) = 0$ for $i = 1, \dots, k$. Hence, $g(v) = \sum_{i=1}^{k} a_i f_i(v) = 0 \implies v \in \ker(g)$.

Proof of $\boxed{\Longleftarrow}$
Proceed by induction on $k \geq 1$.

If $k = 1$, we have $\ker(f_1) \subseteq \ker(g)$. Take $v \notin \ker(f_1)$. Then $V = \ker(f_1) \oplus \operatorname{span}\{v\}$. Define $a_1 = \dfrac{g(v)}{f_1(v)}$. Then $g = a_1 f_1$, because both functionals vanish on $\ker(f_1)$ and $g(v) = a_1 f_1(v)$.

Suppose now the result proven for k: we want to prove it for $k + 1$.

If $\displaystyle\bigcap_{i=1}^{k} \ker(f_i) \subseteq \ker(f_{k+1})$, then we can forget f_{k+1} and use the inductive hypothesis for k to conclude. Otherwise, this means that $\exists v \in \bigcap_{i=1}^{k} \ker(f_i)$ but $v \notin \ker(f_{k+1})$.
Then $\bigcap_{i=1}^{k} \ker(f_i) = \operatorname{span}\{v\} \oplus \bigcap_{j=1}^{k+1} \ker(f_j)$.

Define $a_{k+1} = \dfrac{g(v)}{f_{k+1}(v)}$ and call $g' = g - a_{k+1} f_{k+1}$.

We claim that $\displaystyle\bigcap_{i=1}^{k} \ker(f_i) \subseteq \ker(g')$. In fact, $\bigcap_{j=1}^{k+1} \ker(f_j) \subseteq \ker(g')$ and $g'(v) = 0$.

The conclusion follows applying the inductive hypothesis to $f_1, \dots, f_k, g'$.