18.700 - Fall 2006 - Problem Set 3 (47 points)

(Due on **Tuesday, Oct 3rd**)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". All solutions must be supported by proofs or counterexamples. NO LATE HOMEWORK IS ALLOWED.

Problem 0. (7 points: 3+4)

(a) Is there a homomorphism $A : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ of vector spaces over $\mathbb{R}$ such that $A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ and $A\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$?
If so, determine the matrix associated to A.

(b) Let $L : V \longrightarrow W$ be a homomorphism of vector spaces over a field $\mathbb{F}$. Show that L is surjective if and only if it takes a set of generators for V to a set of generators for W; L is injective if and only if it takes a set of linearly independent vectors in V to a set of linearly independent vectors in W (this "only if" part was done in class); L is invertible if and only if it takes a basis of V to a basis of W.

Problem 1. (10 points: 4+2+4)
Let $\mathbb{R}[t]_{\leq d}$ be the space of polynomials in t of degree at most d with real coefficients. Consider the homomorphism $D : \mathbb{R}[t]_{\leq d} \longrightarrow \mathbb{R}[t]_{\leq d}$ of $\mathbb{R}$-vector spaces defined as $D(q(t)) = q'(t)$, where $q'(t)$ is the derivative of $q(t)$ (remember that the *derivative* of $q(t) = a_0 + a_1 t + \cdots + a_d t^d$ is $q'(t) = a_1 + 2a_2 t + 3a_3 t^2 + \cdots + d a_d t^{d-1}$).

(a) Compute the matrix $M^{\mathcal{B}}_{\mathcal{B}}(D)$ associated to D, where $\mathcal{B} = \{t^k \mid k \leq d\}$ is a basis of $\mathbb{R}[t]_{\leq d}$.

(b) Detemine kernel and image of D.
Is D an isomorphism? If so, compute $M^{\mathcal{B}}_{\mathcal{B}}(D^{-1})$.

(c) Look at the homomorphism of $\mathbb{Z}/p$-vector spaces $\tilde{D} : \mathbb{Z}/p[t] \longrightarrow \mathbb{Z}/p[t]$ given by $\tilde{D}(q(t)) = q'(t)$ as before (but with coefficients in $\mathbb{Z}/p$).
What are kernel and image of $\tilde{D}$?

Problem 2. (12 points: 4+3+5)
Let $A = (a_{ij})_{i,j=1}^n$ be an $n \times n$ square matrix with coefficients in a field $\mathbb{F}$. We say that A is *upper triangular*[1] if $a_{ij} = 0$ whenever $i > j$, and *strictly upper triangular*[1] if $a_{ij} = 0$ whenever $i \geq j$.
Let A and B be two $n \times n$ upper triangular matrices.

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[1]Example: If $A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}$, then $\begin{pmatrix} -1 & 0 & 9 \\ 0 & 7 & -2 \\ 0 & 0 & 0 \end{pmatrix}$ is an upper triangular real 3×3 matrix, which is not strictly upper triangular. On the other hand, $\begin{pmatrix} 0 & 1 & 9 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$ is a strictly upper triangular real 3×3 matrix.

(a) Show that the product $A \cdot B$ is upper triangular.

(b) If A is strictly upper triangular and B is as above, show that $A \cdot B$ and $B \cdot A$ are strictly upper triangular.

(c) Show that, if A is strictly upper triangular, then $A^n = 0$ (i.e. the product of A with itself n times). Show that if A is upper triangular but not strictly, then $A^n \neq 0$ for every integer n.

Problem 3. (8 points)
Let $A \in \mathcal{M}_{m\times n}(\mathbb{F})$ be a matrix with coefficients in $\mathbb{F}$ and let $S_A := \{X \in \mathcal{M}_{n\times k}(\mathbb{F}) \mid A \cdot X = 0\}$. Prove that, for every fixed A, the set S_A is a vector subspace of $\mathcal{M}_{n\times k}(\mathbb{F})$ over $\mathbb{F}$ and compute its dimension (over $\mathbb{F}$).
(Hint: interpret $A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ and $X : \mathbb{F}^k \longrightarrow \mathbb{F}^n$ as homomorphisms so that $A \cdot X : \mathbb{F}^k \longrightarrow \mathbb{F}^m$ is their composition. When is a composition the zero map?)

Problem 4. (10 points: 5+5)

(a) Fix a polynomial $q(s) = a_0 + a_1 s + a_2 s^2 \in \mathbb{R}[s]_{\leq 2}$ in the variable s of degree at most 2 and consider the map (similar to the one given in Problem 3(c) of PSet 2) $\mathrm{ev}_q : \mathbb{R}[t]_{\leq 2} \longrightarrow \mathbb{R}[s]_{\leq 4}$ defined as $\mathrm{ev}_q(p(t)) = p(q(s))$.
Compute the matrix $M^{\mathcal{B}}_{\mathcal{C}}(\mathrm{ev}_q)$ associated to ev_q with respect to the bases $\mathcal{B} = \{v_1 = 1, v_2 = t+1, v_3 = (t+1)^2\}$ and $\mathcal{C} = \{w_1 = 1, w_2 = s, w_3 = s^2, w_4 = s^3, w_5 = s^4\}$.

(b) Let $\mathbb{F}$ be a field and $p \geq 2$ a prime number, and consider the map $Q_p : \mathbb{F}[t] \longrightarrow \mathbb{F}[t]$ defined as $Q_p(q(t)) = q(t)^p$.
Show that, if Q_p is a homomorphism of $\mathbb{F}$-vector spaces, then $\mathbb{F}$ has characteristic p.
Then, assume the fact that Q_p is a homomorphism for $\mathbb{F} = \mathbb{Z}/p$.
Find bases for the kernel and the image of Q_p when $\mathbb{F} = \mathbb{Z}/p$.