18.700 - Fall 2006 - Problem Set 2 (38 points)

(Due on **Tuesday, Sept 26th**)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". All solutions must be supported by proofs or counterexamples. NO LATE HOMEWORK IS ALLOWED.

Problem 1. (4 points)
Let V be a vector space over the field $\mathbb{F}$. Let $W_1, W_2 \subseteq V$ be vector subspaces of V (over $\mathbb{F}$) such that $V = W_1 + W_2$. Show that there exists a vector subspace $U \subseteq W_1$ (over $\mathbb{F}$) such that $V = U \oplus W_2$.

Problem 2. (10 points: 3+3+3+1)
Let $V = \mathbb{C}^4$ as a vector space over $\mathbb{C}$ and let $\mathcal{C} := \{e_1, e_2, e_3, e_4\}$ be the canonical basis of $\mathbb{C}^4$ over $\mathbb{C}$ (where e_k is the vector with all zeroes except a 1 in the k-th position). Consider the subset $\mathcal{B} := \{e_1 - e_2, e_2 - e_3, i(e_3 - e_1), e_3 + e_4\} \subset V$ and call $\text{span}_{\mathbb{C}}(\mathcal{B})$ the span of $\mathcal{B}$ over $\mathbb{C}$ and $\text{span}_{\mathbb{R}}(\mathcal{B})$ the span of $\mathcal{B}$ over $\mathbb{R}$.

(a) What is the dimension of $\text{span}_{\mathbb{C}}(\mathcal{B})$ over $\mathbb{C}$? Find a basis of $\text{span}_{\mathbb{C}}(\mathcal{B})$ over $\mathbb{C}$.

(b) What is the dimension of $\text{span}_{\mathbb{R}}(\mathcal{B})$ over $\mathbb{R}$? Find a basis of $\text{span}_{\mathbb{R}}(\mathcal{B})$ over $\mathbb{R}$.

(c) Complete $\mathcal{B}$ to a basis of $\mathbb{C}^4$ over $\mathbb{R}$.

(d) Find a vector subspace W of $\mathbb{C}^4$ over $\mathbb{C}$ such that $V = W \oplus \text{span}_{\mathbb{C}}(\mathcal{B})$ (i.e. V is the direct sum (over $\mathbb{C}$) of W and $\text{span}_{\mathbb{C}}(\mathcal{B})$).

Problem 3. (18 points: 4+4+6+4)
Let $\mathbb{F}[t] = \{p(t) = a_0 + a_1 t + \cdots + a_d t^d | a_i \in \mathbb{F}\}$ be the set of polynomials in the variable t with coefficients in the field $\mathbb{F}$.

(a) Consider the subspace $W = \{p(t) \in \mathbb{Q}[t] \,|\, p(0) = p(1) = p(2) = 0\}$ over $\mathbb{Q}$ of $\mathbb{Q}[t]$.
Find a *complement* U of W inside $\mathbb{Q}[t]$.
What is the dimension (over $\mathbb{Q}$) of U?

(b) For every polynomial $q(s) \in \mathbb{F}[s]$, show that $\text{ev}_{q(s)} : \begin{array}{ccc} \mathbb{F}[t] & \longrightarrow & \mathbb{F}[s] \\ p(t) & \longmapsto & p(q(s)) \end{array}$ is a homomorphism of vector spaces over $\mathbb{F}$.
[Example: if $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$, then $\text{ev}_{q(s)}(p(t))$ is the polynomial in s obtained by expanding the expression $a_0 + a_1(q(s)) + a_2(q(s))^2 + a_3(q(s))^3$.]

(c) Let $q(t) \in \mathbb{F}[t]$ be a nonzero polynomial.
Show that $\mu_q : \mathbb{F}[t]_{\leq k} \to \mathbb{F}[t]_{\leq k+d}$ that associated to $p(t)$ the product $p(t) \cdot q(t)$ is a homomorphism of vector spaces over $\mathbb{F}$.
Find a subspace $U \subseteq \mathbb{F}[t]_{\leq k+d}$ (over $\mathbb{F}$) such that $\mathbb{F}[t]_{\leq k+d} = U \oplus \mathrm{Im}(\mu_q)$ (over $\mathbb{F}$), where $\mathrm{Im}(\mu_q)$ is the *image* of μ_q.
Compute the dimension of U (over $\mathbb{F}$).

(d) Let $q(t) \in \mathbb{F}[t]$ be a nonzero polynomial. Consider the homomorphism $\mu_q : \mathbb{F}[t] \to \mathbb{F}[t]$ that maps $p(t)$ to the product $p(t) \cdot q(t)$, as in (c).
Find a subspace $U \subseteq \mathbb{F}[t]$ (over $\mathbb{F}$) such that $\mathbb{F}[t] = U \oplus \mathrm{Im}(\mu_q)$ (over $\mathbb{F}$).
Compute the dimension on U (over $\mathbb{F}$).

Problem 4. (6 points: 3+3)

(a) Let $v \in \mathbb{R}^3$ be a nonzero vector. Show that $\Pi_v : \begin{array}{rcl} \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ w & \longmapsto & v \times w \end{array}$ is a homomorphism of vector spaces over $\mathbb{R}$, where $v \times w$ is the vector product of v and w.
Determine the *kernel* and the image of Π_v.

(b) Let $\theta \in [0, 2\pi)$ be an angle. Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the map that associates to a vector v the vector $R_\theta(v)$ obtained performing a (counterclockwise) rotation centered at the origin of angle θ.
Show that R_θ is a homomorphism of vector spaces over $\mathbb{R}$.
Determine kernel and image of $\mathbb{R}_\theta$.

If V and W are vector spaces over a field $\mathbb{F}$ and $f : V \to W$ is a homomorphism of vector spaces over $\mathbb{F}$, then the **kernel** *of f is defined as $\ker(f) := \{v \in V \mid f(v) = \vec{0} \in W\} \subseteq V$ and the* **image** *of f is defined as $\mathrm{Im}(f) := \{w \in W \mid \exists v \in V : f(v) = w\} \subseteq W$.*

Let $p(t)$ be a nonzero polynomial with coefficients in a field $\mathbb{F}$. We can write $p(t) = a_0 + a_1 t + \cdots + a_d t^d$ with $a_d \neq 0$, for suitable d and $a_0, a_1, \ldots, a_d \in \mathbb{F}$. Define the **degree** *of $p(t)$ to be d.*

Let V be a vector space over the field $\mathbb{F}$ and let $W \subseteq V$ be a vector subspace of V over $\mathbb{F}$. A **complement** *of W inside V over the field $\mathbb{F}$ is a subspace $U \subseteq V$ over $\mathbb{F}$ such that $V = U \oplus W$ over $\mathbb{F}$.*