18.700 - Fall 2006 - Solutions to Problem Set 2

Problem 1.

Define $Z := W_1 \cap W_2$ to be the intersection of W_1 and W_2. It is a vector subspace of W_1 (and also of W_2 and of V) over $\mathbb{F}$. Pick a basis $\mathcal{B}_Z$ of Z. Complete it to a basis $\mathcal{B}_1$ of W_1. Define

$$U := \operatorname{span}_{\mathbb{F}}(\mathcal{B}_1 \setminus \mathcal{B}_Z)$$

and call $\mathcal{B}_U := \mathcal{B}_1 \setminus \mathcal{B}_Z$, which is thus a basis of U (over $\mathbb{F}$).
$U \cap Z = \{\vec{0}\}$ (why?) and so $U \cap W_2 = \{\vec{0}\}$.
To show that $U + W_2 = V$, complete $\mathcal{B}_Z$ to a basis of W_2.
As $W_1 + W_2 = V$, then every $v \in V$ can be written as

$$v = (a_1u_1 + \cdots + a_nu_n) + (b_1z_1 + \cdots + b_mz_m) + (c_1v_1 + \cdots + c_lv_l)$$

where $a_i, b_j, c_k \in \mathbb{F}$, $u_i \in \mathcal{B}_U$, $z_j \in \mathcal{B}_Z$ and $v_k \in \mathcal{B}_2$.
Then $u = (a_1u_1 + \cdots + a_nu_n) \in U$, $w_2 = (b_1z_1 + \cdots = b_mz_m) + (c_1v_1 + \cdots + c_lv_l) \in W_2$ and $v = u + w_2$.

Problem 2.

(a) The set $\widetilde{\mathcal{B}} = \{e_1 - e_2, e_2 - e_3, e_3 + e_4\}$ generates $\mathcal{B}$ over $\mathbb{C}$ (and so $\operatorname{span}_{\mathbb{C}}(\mathcal{B})$ over $\mathbb{C}$) because $i(e_3 - e_1) = -i(e_1 - e_2) - i(e_2 - e_3)$.
Moveover, $\widetilde{\mathcal{B}}$ is a set of linearly independent vectors over $\mathbb{C}$, because $a(e_1 - e_2) + b(e_2 - e_3) + c(e_3 + e_4) = ae_1 + (b-a)e_2 + (c-b)e_3 + ce_4 = \begin{pmatrix} a \\ b-a \\ c-b \\ c \end{pmatrix}$ with $a, b, c \in \mathbb{C}$ is zero if and only if $a = b = c = 0$.
Thus, $\widetilde{\mathcal{B}}$ is a basis of $\operatorname{span}_{\mathbb{C}}(\mathcal{B})$ and so $\dim_{\mathbb{C}} \operatorname{span}_{\mathbb{C}}(\mathcal{B}) = 3$.

(b) $\mathcal{B}$ is a set of linearly independent vectors over $\mathbb{R}$. In fact, $a(e_1 - e_2) + b(e_2 + e_3) + c[i(e_3 - e_1)] + d(e_3 + e_4) = \begin{pmatrix} a - ic \\ b - a \\ ic + d \\ d \end{pmatrix}$ with $a, b, c, d \in \mathbb{R}$ is zero if and only if $a = b = c = d = 0$.
Thus, $\mathcal{B}$ is a basis of $\operatorname{span}_{\mathbb{R}}(\mathcal{B})$ over $\mathbb{R}$ and $\dim_{\mathbb{R}} \operatorname{span}_{\mathbb{R}}(\mathcal{B}) = 4$.

(c) $\dim_{\mathbb{R}}(\mathbb{C}^4) = 2 \cdot 4 = 8$, so we need to add other 4 vectors to $\mathcal{B}$.
Claim: $\mathcal{B}' := \mathcal{B} \cup \{e_1, ie_2, ie_3, ie_4\}$ is a basis of $\mathbb{C}^4$ over $\mathbb{R}$.
We only need that they generate $\{e_1, ie_1, e_2, ie_2, e_3, ie_3, e_4, ie_4\}$ over $\mathbb{R}$.

$$\begin{array}{cc}
(e_1) \in \mathcal{B}' & (ie_1) = -[i(e_3 - e_1)] + (ie_3) \\
e_2 = -(e_1 - e_2) + (e_1) & (ie_2) \in \mathcal{B}' \\
e_3 = -(e_1 - e_2) - (e_2 - e_3) + (e_1) & (ie_3) \in \mathcal{B}' \\
e_4 = (e_1 - e_2) + (e_2 - e_3) + (e_3 + e_4) - (e_1) & (ie_4) \in \mathcal{B}'
\end{array}$$

(d) From (a) we have that $\widetilde{\mathcal{B}} = \{e_1 - e_2, e_2 - e_3, e_3 + e_4\}$ is a basis of $\operatorname{span}_{\mathbb{C}}(\mathcal{B})$ over $\mathbb{C}$.
Define $W := \operatorname{span}_{\mathbb{C}}\{e_1\}$. If $a, b, c, d \in \mathbb{C}$, then $a(e_1 - e_2) + b(e_2 - e_3) + c(e_3 + e_4) = d(e_1)$ implies $a = b = c = d = 0$. This shows that $\widetilde{\mathcal{B}}$ is a set of linearly independent vectors and that $W \cap \operatorname{span}_{\mathbb{C}}(\mathcal{B}) = \{\vec{0}\}$. As $|\widetilde{\mathcal{B}}| = 4 = \dim_{\mathbb{C}} V$, then $\widetilde{\mathcal{B}}$ is a basis of V over $\mathbb{C}$ and so $W + \operatorname{span}_{\mathbb{C}}(\mathcal{B}) = V$. Hence, $V = W \oplus \operatorname{span}_{\mathbb{C}}(\mathcal{B})$.

Problem 3.

(a) Define $U := \mathbb{Q}[t]_{\leq 2} = \mathrm{span}_{\mathbb{Q}}\{1, t, t^2 - t\}$. Then $U \cap W = \{0\}$ because, if $p(t) \in \mathbb{Q}[t]$ has degree at most 2 and vanishes at three points (in our case, at $0, 1, 2$), then $p(t) = 0$.
We want to show that $\mathbb{Q}[t] = U + W$. For every $p(t) \in \mathbb{Q}[t]$, define $a_0 := p(0)$, $a_1 := p(1) - p(0)$ and $a_2 := p(2)/2 - p(1) + p(0)/2$, and $s(t) = a_0 \cdot 1 + a_1 \cdot t + a_2 \cdot (t^2 - t) \in U$. Then $p(t) = s(t) + [p(t) - s(t)]$. We want to show that $p(t) - s(t) \in W$. Notice that $s(0) = p(0)$, $s(1) = p(0) + [p(1) - p(0)] = p(1)$ and $s(2) = p(0) + 2[p(1) - p(0)] + [p(2)/2 - p(1) + p(0)/2](2^2 - 2) = p(0) + 2p(1) - 2p(0) + p(2) - 2p(1) + p(0) = p(2)$. Hence, $(p - s)(0) = (p - s)(1) = (p - s)(2) = 0$ and so $p(t) - s(t) \in W$.
Clearly, $\{1, t, t^2 - t\}$ are linearly independent (over $\mathbb{Q}$) and so they are a basis of U. Hence, $\dim_{\mathbb{Q}} U = 3$.

(b) We have to show that $\mathrm{ev}_{q(s)}$ respects sums and scalar multiplication.
If $p(t), r(t) \in \mathbb{F}[t]$, then $\mathrm{ev}_{q(s)}(p + r) = (p + r)(q(s)) = p(q(s)) + r(q(s)) = \mathrm{ev}_{q(s)}(p) + \mathrm{ev}_{q(s)}(r)$.
If $\lambda \in \mathbb{F}$ and $p(t) \in \mathbb{F}[t]$, then $\mathrm{ev}_{q(s)}(\lambda \cdot p) = (\lambda \cdot p)(q(s)) = \lambda \cdot p(q(s)) = \lambda \cdot \mathrm{ev}_{q(s)}(p)$.

(c) We have to show that μ_q respects sums and scalar multiplication.
If $p(t), r(t) \in \mathbb{F}[t]_{\leq k}$, then $\mu_q(p(t) + r(t)) = (p(t) + r(t))q(t) = p(t)q(t) + r(t)q(t) = \mu_q(p(t)) + \mu_q(r(t))$.
If $p(t) \in \mathbb{F}[t]_{\leq k}$ and $\lambda \in \mathbb{F}$, then $\mu_q(\lambda p(t)) = \lambda p(t) r(t) = \lambda \mu_q(p(t)$.
Define $U := \mathbb{F}[t]_{\leq d-1} = \mathrm{span}_{\mathbb{F}}\{t^i \mid 0 \leq i \leq d - 1\}$.
Clearly, $\mathrm{Im}(\mu_q) \cap U = \{0\}$ because every polynomial in $\mathrm{Im}(\mu_q)$ is a multiple of $q(t)$ and is either 0 or it has degree $\geq d$.
We want to show that $\mathcal{B}_k = \{1, t, t^2, \ldots, t^{d-1}, \mu_q(1), \mu_q(t), \ldots, \mu_q(t^k)\}$ is a basis of $\mathbb{F}[t]_{k+d}$ over $\mathbb{F}$. As $|\mathcal{B}_k| = k + d + 1 = \dim_{\mathbb{F}} \mathbb{F}[t]_{k+d}$, we only need to show that they are linearly independent. This is clear, because $\mu_q(t^n)$ has degree $d + n$, so they all have different degrees.
Hence, $U \oplus \mathrm{Im}(\mu_q) = \mathbb{F}[t]_{\leq k+d}$ and $\dim_{\mathbb{F}} U = \dim_{\mathbb{F}} \mathbb{F}[t]_{\leq d-1} = d$.

(d) Define $U := \mathbb{F}[t]_{\leq d-1} = \mathrm{span}_{\mathbb{F}}\{t^i \mid 0 \leq i \leq d - 1\}$ as in (c).
Again, $\mathrm{Im}(\mu_q) \cap U = \{0\}$ because every polynomial in $\mathrm{Im}(\mu_q)$ is a multiple of $q(t)$ and is either 0 or it has degree $\geq d$.
As in (*c*), the set $\mathcal{B} = \{t^i \mid 0 \leq i \leq d - 1\} \cup \{\mu_q(t^j) \mid j \geq 0\}$ is made of polynomials of distinct degree, so it is a set of linearly independent vectors.
To show that $\mathcal{B}$ generates, pick a nonzero polynomial $p(t) \in \mathbb{F}[t]$. If $\deg(p(t)) < d$, then we are done.
Otherwise, observe that $p(t) \in \mathbb{F}[t]_{\leq k+d}$ for k large enough (for instance, every $k \geq (\deg p) - d$ works). Because of (c), $p(t)$ can be written as a linear combination of polynomials in $\mathcal{B}_k \subset \mathcal{B}$.
As in (c), $\dim_{\mathbb{F}} U = d$.

Problem 4. (5 points: 2+3)

(a) Π_v respects sums, because $\Pi_v(w_1 + w_2) = v \times (w_1 + w_2) = v \times w_1 + v \times w_2 = \Pi_v(w_1) + \Pi_v(w_2)$ for all $w_1, w_2 \in \mathbb{R}^3$.
Π_v respects scalar multiplication, because $\Pi_v(\lambda \cdot w) = v \times (\lambda w) = \lambda(v \times w) = \lambda \cdot \Pi_v(w)$ for all $\lambda \in \mathbb{R}$ and $w \in \mathbb{R}^3$.
Hence, Π_v is a homomorphism of vector spaces over $\mathbb{R}$.
The kernel $\ker\Pi_v = \mathrm{span}_{\mathbb{R}}\{v\}$, because $\Pi_v(w) = \vec{0} \iff v \times w = \vec{0} \iff w$ is a multiple of v.
The image $\mathrm{Im}\Pi_v = \{w \in \mathbb{R}^3 \,|\, w \text{ is orthogonal to } v\}$.

(b) Given a vector $v = \begin{pmatrix} x \\ y \end{pmatrix}$, we have $R_\theta(v) = \begin{pmatrix} x\cos(\theta) - y\sin(\theta) \\ x\sin(\theta) + y\cos(\theta) \end{pmatrix}$.
Hence, it is straightforward to check that $R_\theta(v + w) = R_\theta(v) + R_\theta(w)$ for $v, w \in \mathbb{R}^2$ and that $R_\theta(\lambda v) = \lambda R_\theta(v)$ for $v \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$. Thus, R_θ is a homomorphism of vector spaces over $\mathbb{R}$.
$R_\theta(v) = \vec{0} \implies v = \vec{0}$, so that $\ker R_\theta = \{\vec{0}\}$.
Instead, $\mathrm{Im}(R_\theta) = \mathbb{R}^2$. In fact, given $v \in \mathbb{R}^2$ and called $w := R_{2\pi-\theta}(v)$, we have $R_\theta(w) = v$.