18.700 - Fall 2006 - Problem Set 1 (28 points)

(Due Thursday, Sept 14th)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". All solutions must be supported by proofs or counterexamples. NO LATE HOMEWORK IS ALLOWED.

Problem 1. (5 points: 2+3)

Let $\mathbb{Q}(\sqrt{2}) = \left\{ a + b\sqrt{2} \in \mathbb{R} \,\middle|\, a, b \in \mathbb{Q} \right\}$, i.e. the subset of the real numbers that can be written as $a + b\sqrt{2}$, with a, b rational numbers.

(a) Is $\mathbb{Q}(\sqrt{2})$ a subfield of $\mathbb{R}$?

(b) Find a basis for $\mathbb{Q}(\sqrt{2})$ as a vector space over the field $\mathbb{Q}$.

Problem 2. (8 points: 2+2+2+2)

Let k be a complex number. Consider the subsets $V_k := \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2 \,\middle|\, z_1 + z_2 = k \right\} \subset \mathbb{C}^2$ and $W_k := \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2 \,\middle|\, z_1 + \bar{z}_2 = k \right\} \subset \mathbb{C}^2$, where $\bar{z}_2$ is the complex conjugate of z_2.

(a) For which values of $k \in \mathbb{C}$ is V_k a vector subspace of $\mathbb{C}^2$ over $\mathbb{C}$?

(b) For the values of k found in (a), find a basis of V_k over $\mathbb{C}$.

(c) For which values of $k \in \mathbb{C}$ is W_k a vector subspace of $\mathbb{C}^2$ over $\mathbb{C}$?

(d) For which values of $k \in \mathbb{C}$ is W_k a vector subspace of $\mathbb{C}^2$ over $\mathbb{R}$?

Problem 3. (10 points: 2+2+3+3)

Let $\mathbb{R}[t]$ be the set of polynomials in t with real coefficients, that is polynomials that look like $p(t) = a_d t^d + a_{d-1} t^{d-1} + \cdots + a_2 t^2 + a_1 t + a_0$, where the a_i's are real numbers.

(a) Let $\mathbb{R}[t]_d \subset \mathbb{R}[t]$ be the subset of polynomials of degree *exactly* equal to d.
Is it a (vector) subspace of $\mathbb{R}[t]$ over $\mathbb{R}$? If so, find a basis.

(b) For every $k \in \mathbb{R}$, define $E_k := \{p(t) \in \mathbb{R}[t] \,|\, p(2) = k\}$, that is the subset of polynomials in $\mathbb{R}[t]$ that take the value k at 2.
For which values of k is E_k a (vector) subspace of $\mathbb{R}[t]$ over $\mathbb{R}$?

(c) For the values of k found in (b), is $E_k \cap \mathbb{R}[t]_{\leq d}$ (i.e. the subset of polynomials of degree at most d that take the value k at 2) a vector subspace of $\mathbb{R}[t]$ over $\mathbb{R}$? If so, find a basis of $E_k \cap \mathbb{R}[t]_{\leq 2}$ over $\mathbb{R}$.

(d) For the values of k found in (b), find a basis of E_k.

Problem 4. (5 points: 2+3)

(a) What is the maximum number of nonzero vectors in $(\mathbb{Z}/3\mathbb{Z})^2$ that are *pairwise linearly independent*?

(b) Let p be a prime number and let $n \geq 1$ be an integer. What is the maximum number of nonzero vectors in $(\mathbb{Z}/p\mathbb{Z})^n$ that are pairwise linearly independent?
(Hint: consider the nonzero vectors of $(\mathbb{Z}/p\mathbb{Z})^n$ and say that two vectors belong to the same *class* if they are proportional. The question asks: how many classes are there? It may help to answer first to the following question: how many vectors belong to the same class?)