18.700 - Fall 2006 - Practice Exam B

Problem 1.

(a) Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation around the origin of an angle θ. Is there a basis $\mathcal{B}$ of $\mathbb{R}^2$ such that $M_{\mathcal{B}}^{\mathcal{B}}(R_\theta)$ is diagonal? And upper triangular?

(b) Consider the matrix with real coefficients associated to R_θ as in (a). As $\mathbb{R} \subset \mathbb{C}$, the same matrix defines a homomorphism $R_\theta^{\mathbb{C}} : \mathbb{C}^2 \to \mathbb{C}^2$. Is there a basis $\mathcal{C}$ of $\mathbb{C}^2$ such that $M_{\mathcal{C}}^{\mathcal{C}}(R_\theta)$ is diagonal?

(c) Let $R_\theta : \mathbb{R}^3 \to \mathbb{R}^3$ be the rotation around the axis e_1 of an angle θ. Is there a basis $\mathcal{D}$ of $\mathbb{R}^3$ such that $M_{\mathcal{D}}^{\mathcal{D}}(R_\theta)$ is diagonal or upper triangular?

Problem 2.

Let $L : \mathbb{Q}^3 \longrightarrow \mathbb{Q}^3$ be the homomorphism of $\mathbb{Q}$-vector spaces defined by the matrix $\begin{pmatrix} -1 & 1 & -1 \\ -2 & 5 & 1 \\ 1 & 3 & 5 \end{pmatrix}$.

Find bases $\mathcal{B}$ and $\mathcal{C}$ in such a way that: there exists a k such that $M_{\mathcal{C}}^{\mathcal{B}}(L)_{ij} = 1$ for $1 \leq i = j \leq k$ and $M_{\mathcal{C}}^{\mathcal{B}}(L)_{ij} = 0$ otherwise[1].

Problem 3.

Let V, W be vector spaces over $\mathbb{F}$ and V^*, W^* their dual. Let $L : V \longrightarrow W$ be a homomorphism and let $L^* : W^* \longrightarrow V^*$ be its dual.

Show that $\mathrm{Ann}_V(\ker(L)) = \mathrm{Im}(L^*)$ and $\mathrm{Ann}_W(\mathrm{Im}(L)) = \ker(L^*)$.

[*Remember that, if $S \subseteq V$ is a subset, the* **annihilator** $\mathrm{Ann}_V(S) = \{\varphi \in V^* \mid \varphi(s) = 0 \ \forall s \in S\} \subseteq V^*$.]

Problem 4.

Let $V = \mathbb{R}[t]$ the real vector space of polynomials in the variable t. For every $x \in \mathbb{R}$, let $\mathrm{ev}_x : V \longrightarrow \mathbb{R}$ be the evaluation at x, defined as usual: $\mathrm{ev}_x(p(t)) = p(x)$.

Is $\mathcal{B} = \{\mathrm{ev}_x \in V^* \mid x \in \mathbb{R}\}$ is set of generators for V^*?

Is $\mathcal{B}$ a set of linearly independent vectors in V^*?

What happens if we replace the field $\mathbb{R}$ with the field $\mathbb{Z}/p$ (where p is a prime)?

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[1]For 3×3 matrices, this means that $M_{\mathcal{C}}^{\mathcal{B}}(L)$ must look like: either $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, or $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$