18.700 - Fall 2006 - Practice Exam B

Problem 1.

(a) Such a basis exists only if $\theta = 0$ or $\theta = \pi$.

Suppose there is a basis $\mathcal{B} = \{v_1, v_2\}$ of $\mathbb{R}^2$ such that $M_{\mathcal{B}}^{\mathcal{B}}(R_\theta)$ is upper triangular.
Then $R_\theta(v_1) = \lambda_1 v_1$, where λ_1 is $M_{\mathcal{B}}^{\mathcal{B}}(R_\theta)_{11}$. As R_θ is a rotation, the only possible values for λ_1 are 1 and -1.
If $\lambda_1 = 1$, then $M_{\mathcal{B}}^{\mathcal{B}}(R_\theta) = I_2$ the identity matrix, and so R_θ is the identity (and so $M_{\mathcal{B}}^{\mathcal{B}}(R_\theta)$ is the identity for every $\mathcal{B}$).
If $\lambda_1 = -1$, then $M_{\mathcal{B}}^{\mathcal{B}}(R_\theta) = -I_2$ and so $\theta = \pi$ (and $M_{\mathcal{B}}^{\mathcal{B}}(R_\theta)$ is $-I_2$ for every $\mathcal{B}$).
If $\theta \neq 0, \pi$, then $M_{\mathcal{B}}^{\mathcal{B}}(R_\theta)$ is never upper triangular (and so never diagonal).

(b) Yes, an example of such a basis is $\mathcal{C} = \{v_1 = e_1 + ie_2, v_2 = e_1 - ie_2\}$ (the other bases look like $\{\alpha_1 v_1, \alpha_2 v_2\}$ with $0 \neq \alpha_1, \alpha_2 \in \mathbb{C}$).
In fact, $R_\theta(e_1) = \cos\theta e_1 + \sin\theta e_2$ and $R_\theta(e_2) = -\sin\theta e_1 + \cos\theta e_2$.
So, $R_\theta(e_1 + ie_2) = (\cos\theta - i\sin\theta)(e_1 + ie_2)$ and $R_\theta(e_1 - ie_2) = (\cos\theta + i\sin\theta)(e_1 - ie_2)$.

(c) Such a basis exists only if $\theta = 0$ or $\theta = \pi$.
If $\theta = 0$, then R_θ is the identity and so $M_{\mathcal{B}}^{\mathcal{B}}(R_\theta) = I_3$ for every basis $\mathcal{B}$.
If $\theta = \pi$, then we can choose v_1 to be a nonzero vector along the axis of rotation and v_2, v_3 to be a basis of the plane perpendicular to the axis. Then we obtain $M_{\mathcal{B}}^{\mathcal{B}}(R_\pi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, which is diagonal.

Assume now that $\theta \neq 0, \pi$ and suppose there exists a basis $\mathcal{B} = \{v_1, v_2, v_3\}$ of $\mathbb{R}^3$ such that $M_{\mathcal{B}}^{\mathcal{B}}(R_\theta)$ is upper triangular. Then $R_\theta(v_1)$ is a multiple of v_1, and hence it must be a vector along the axis of rotation.
Because R_θ is a rotation, it preserves the plane Π perpendicular to v_1, i.e. $R_\theta(w) \in \Pi$ for every $w \in \Pi$. Moreover, this is the only plane[1] preserved by R_θ, because $\theta \neq 0, \pi$.
As we are assuming that $M_{\mathcal{B}}^{\mathcal{B}}(R_\theta)$ is upper triangular, then the plane spanned by v_1 and v_2 is also preserved. This implies that $\Pi = \operatorname{span}\{v_1, v_2\}$, which is clearly a contradiction.
Hence, if $\theta \neq 0, \pi$, there is no basis $\mathcal{B}$ such that $M_{\mathcal{B}}^{\mathcal{B}}(R_\theta)$ is upper triangular.

Problem 2.

Define $\mathcal{B} = \left\{ v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\}$ and
$\mathcal{C} = \left\{ w_1 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}, w_2 = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}, w_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.
You can easily check that both $\mathcal{B}$ and $\mathcal{C}$ are bases of $\mathbb{Q}^3$.
A straighforward computation gives $L(v_1) = L(e_1) = w_1$, $L(v_2) = L(e_2) = w_2$ and $L(v_3) = 0$.

[1] I know that this requires some justification. Try to figure out why it is so.

Hence, with the choices of $\mathcal{B}$ and $\mathcal{C}$ made above, $M^{\mathcal{B}}_{\mathcal{C}}(L) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

(The choices of $\mathcal{B}$ and $\mathcal{C}$ made above work, but they are not unique: you can find other bases that work as well.)

Problem 3.

Proof that $\mathrm{Ann}_V(\ker(L)) \supseteq \mathrm{Im}(L^*)$.
If $\varphi \in \mathrm{Im}(L^*)$, then there exists $\psi \in W^*$ such that $\varphi = L^*(\psi) = \psi \circ L$. Hence, for every $v \in \ker(L)$, we have $\varphi(v) = \psi \circ L(v) = 0 \implies \varphi \in \mathrm{Ann}_V(\ker(L))$.

Proof that $\mathrm{Ann}_V(\ker(L)) \subseteq \mathrm{Im}(L^*)$.
If $\varphi \in \mathrm{Ann}_V(\ker(L))$, then $\varphi(v) = 0\ \forall v \in \ker(L)$.
We want to find $\psi \in W^*$ such that $\varphi = L^*(\psi) = \psi \circ L$.
Let $U \subseteq W$ be a complement of $\mathrm{Im}(L)$, i.e. $W = U \oplus \mathrm{Im}(L)$. To define $\psi(w)$ for every $w \in W$, rewrite $w = u + L(v)$, with $u \in U$ and $v \in V$ *(warning: here the choice of $v \in V$ in not unique, which means that later we have to check that ψ is well-defined)*, and define $\psi(w) = \psi(u + L(v)) = \varphi(v)$.
Suppose now that $w = u + L(v')$, with $v' \in V$. As $L(v') = w - u = L(v)$, we have $L(v' - v) = 0$ and so $v' - v = k \in \ker(L)$. Hence, $\psi(w) = \varphi(v) = \varphi(v' - k) = \varphi(v') - \varphi(k) = \varphi(v')$ and so ψ is well-defined (i.e. independent of the choice of v).
Clearly, $L^*(\psi)(v) = \psi \circ L(v) = \varphi(v)$ for every $v \in V$.

Proof that $\mathrm{Ann}_W(\mathrm{Im}(L)) = \ker(L^*)$.
$\psi \in \ker(L^*) \iff L^*(\psi) = \psi \circ L = 0 \iff \psi \circ L(v) = 0\ \forall v \in V \iff \psi(w) = 0\ \forall w \in \mathrm{Im}(L) \iff \psi \in \mathrm{Ann}_W(\mathrm{Im}(L))$.

Problem 4.

Proof that $\mathcal{B}$ is a set of linear independent vectors on $\mathbb{R}$.
Let $a_1 \mathrm{ev}_{x_1} + \cdots + a_n \mathrm{ev}_{x_n} = 0$ (this is an equality in V^*) for $a_1, \ldots, a_n \in \mathbb{R}$ and $x_1, \ldots, x_n \in \mathbb{R}$ (notice that $x_1, \ldots, x_n$ are distinct).
If we evaluate the equality above at the vector $(t - x_2)(t - x_3) \cdots (t - x_n)$, we obtain $a_1 = 0$. Evaluating at the vector $(t - x_1)(t - x_3) \cdots (t - x_n)$, we obtain $a_2 = 0$. Similarly, we obtain that $a_1 = a_2 = \cdots = a_n = 0$.

Counterexample: $\mathcal{B}$ is **not** *a set of generators.*
Consider the functional $\varphi : V = \mathbb{R}[t] \longrightarrow \mathbb{R}$ defined as $\varphi(t) = 1$ and $\varphi(t^d) = 0$ for $d \neq 1$.
We want to show that $\varphi \notin \mathrm{span}(\mathcal{B})$.
For every $\lambda \in \mathbb{R}$, define the homomorphism $L_\lambda : V \longrightarrow V$ as $L_\lambda(p(t)) = p(\lambda t)$.
The dual homomorphism $L^*_\lambda : V^* \longrightarrow V^*$ has the property that $L^*(\mathrm{ev}_x) = \mathrm{ev}_x \circ L_\lambda = \mathrm{ev}_{\lambda x}$ for every $x \in \mathbb{R}$. Moreover, $L^*(\varphi) = \varphi \circ L = \lambda\varphi$.
By contradiction, suppose that $\varphi = a_1 \mathrm{ev}_{x_1} + \cdots + a_n \mathrm{ev}_{x_n}$ for some $a_1, \ldots, a_n \in \mathbb{R}$ and $x_1, \ldots, x_n \in \mathbb{R}$. We can assume that $a_1, \ldots, a_n \neq 0$ and that the x_i's are all distinct.
Then $\lambda\varphi = L^*(\varphi) = L^*(a_1 \mathrm{ev}_{x_1} + \cdots + a_n \mathrm{ev}_{x_n}) = a_1 \mathrm{ev}_{\lambda x_1} + \cdots + a_n \mathrm{ev}_{\lambda x_n} \implies$
$\implies \lambda a_1 \mathrm{ev}_{x_1} + \cdots + \lambda a_n \mathrm{ev}_{x_n} = a_1 \mathrm{ev}_{\lambda x_1} + \cdots + a_n \mathrm{ev}_{\lambda x_n}$.
If we choose $0 \neq \lambda \in \mathbb{R}$ such that the set $\{x_1, \ldots, x_n\}$ is different from the set $\{\lambda x_1, \ldots, \lambda x_n\}$, then the equality gives a contradiction, because $\mathcal{B}$ is a set of linearly independent vectors (as shown above).

The case of $V = (\mathbb{Z}/p)[t]$.
The same proof as for $\mathbb{R}$ shows that $\mathcal{B}$ is a (finite) set of linearly independent vectors.
Clearly, $\mathcal{B}$ cannot be a set of generators for V^*, because it is a finite set, while V^* has infinite dimension.

(Notice that the conclusion is the same as for $\mathbb{R}$*, but the proof that* $\mathcal{B}$ *is not a set of generators is different. In the case of* $\mathbb{Z}/p$*, we are exploiting the fact that* $\mathcal{B}$ *is a finite set, which depends on the fact that* $\mathbb{Z}/p$ *is a finite field. In the other case, we are exploiting the fact that* $\mathbb{R}$ *is an infinite field, exactly in the part of the proof when we say: given a finite set* $\{x_1, \ldots, x_n\} \subset \mathbb{R}$*, we can find* $0 \neq \lambda \in \mathbb{R}$ *such that* $\{x_1, \ldots, x_n\} \neq \{\lambda x_1, \ldots, \lambda x_n\}$*.*
The difference in the proofs depends essentially on the finiteness/infiniteness of the field.)