18.700 - Fall 2006 - Practice Exam A

Problem 1.

Let $L : \mathbb{C}^4 \longrightarrow \mathbb{C}^3$ be the $\mathbb{C}$-linear homomorphism given by the matrix $\begin{pmatrix} 1 & 1 & 0 & 1 \\ -2 & 0 & 1 & -1 \\ 1 & 3 & 1 & 2 \end{pmatrix}$.

Find bases (over $\mathbb{C}$) for $\ker(L)$ and $\mathrm{Im}(L)$.

For every value of $c \in \mathbb{C}$, find the set of solutions of the linear system $L \cdot X = \begin{pmatrix} 2 \\ 0 \\ c \end{pmatrix}$.

Problem 2.

Let $L : V \longrightarrow W$ be a homomorphism of vector spaces (over some fixed field $\mathbb{F}$).
Show that, if L is injective, then there exists a vector space U and a homomorphism $F : U \longrightarrow W$ such that $\tilde{L} : V \times U \longrightarrow W$ defined as $\tilde{L}(v, u) = L(v) + F(u)$ is an isomorphism.
Similarly, show that, if L is surjective, then there exists a vector space Z and a homomorphism $G : V \longrightarrow Z$ such that the homomorphism $L' : V \longrightarrow W \times Z$ defined as $L'(v) = (L(v), G(v))$ is an isomorphism.
[*Given two vector spaces* V, U *over* $\mathbb{F}$*, the structure of* $\mathbb{F}$*-vector space on the* **product** $V \times U := \{(v, u) \mid v \in V, \quad u \in U\}$ *is given by componentwise sum and scalar multiplication.*]

Problem 3.

Let $A, B : \mathbb{F}^n \longrightarrow \mathbb{F}^n$ and suppose that A is a diagonal matrix with $A_{ii} = \lambda_i$ and assume that $\lambda_i \neq \lambda_j$ if $i \neq j$. Show that, if $AB = BA$, then B is also a diagonal matrix.
Is this still true if we do not assume that the λ's are all distinct?

Problem 4.

Let $V = \mathbb{C}[t]_{\leq 2}$ be the vector space of polynomials of degree at most 2 in t and complex coefficients and let $\mathcal{B}^* = \{\mathrm{ev}_0, \mathrm{ev}_1, \mathrm{ev}_i\}$ be a basis of V^* (where ev_x is the linear functional that evaluates the polynomial at $x \in \mathbb{C}$).
What is the basis $\mathcal{B}$ of $V = V^{**}$ dual to $\mathcal{B}^*$?
Determine the matrix $M^{\mathcal{B}^*}_{\mathcal{B}^*}(L^*)$ if $L : V \longrightarrow V$ is the homomorphism defined as $L(p(t)) = t \cdot p'(t)$ (where $p'(t)$ is the derivative of $p(t)$).