18.700 - Fall 2006 - Solutions of Practice Exam A

Problem 1.

$L(e_i) \in \mathbb{C}^3$ is the i-th column of L. Call it v_i.

Notice that $2v_3 = v_2 - v_1$ and $v_4 = v_2 - v_3 = v_2 - \frac{v_2 - v_1}{2} = \frac{v_1 + v_2}{2}$. Hence, $\mathrm{Im}(L)$ is spanned by v_1, v_2. Moreover, $v_1, v_2 \neq 0$ and they are not proportional. This shows that $\mathcal{B} = \{v_1, v_2\} \subset \mathbb{C}^3$ is a set of linearly independent vectors and so a basis of $\mathrm{Im}(L)$.

As $\dim\ker(L) + \dim\mathrm{Im}(L) = \dim\mathbb{C}^4 = 4$, then we expect a bidimensional $\ker(L)$.

As $2v_3 = v_2 - v_1$ and $v_4 = v_2 - v_3$, we have $L\begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix} = 0$ and $L\begin{pmatrix} 0 \\ 1 \\ -1 \\ -1 \end{pmatrix} = 0$.

$\mathcal{C} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ -1 \end{pmatrix} \right\} \subset \mathbb{C}^4$ is a basis of $\ker(L)$, because it is a maximal set of linearly independent vectors (as they are nonzero and nonproportional).

The system has solution if and only if $B = \begin{pmatrix} 2 \\ 0 \\ c \end{pmatrix}$ belongs to $\mathrm{Im}(L) = \mathrm{span}\{v_1, v_2\}$. Looking at the second entry of the vectors, we conclude that $B \in \mathrm{Im}(L)$ if and only if B is a multiple of v_2, i.e. if and only if $c = 6$.

In this case, $L(e_2) = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix}$, so e_2 is a particular solution of the system and the set of solutions is

$$e_2 + \ker(L) = \{e_2 + v \in \mathbb{C}^4 \,|\, v \in \ker(L)\} = \left\{ \begin{pmatrix} a \\ 1 - a + b \\ 2a - b \\ -b \end{pmatrix} \in \mathbb{C}^4 \,\middle|\, \forall a, b \in \mathbb{C} \right\}.$$

Problem 2.

Let L be injective and define U to be a subspace of W which is complementary to $\mathrm{Im}(L)$, i.e. $U \subseteq W$ and $W = \mathrm{Im}(L) \oplus U$. Define $F : U \longrightarrow W$ to be the inclusion homomorphism.

Let's show that, with the choices made above, $\tilde{L}$ is an isomorphism.

To show that $\tilde{L}$ is injective, let $(v, u) \in V \times U$ be a vector such that $\tilde{L}(v, u) = 0$. We want to prove that $(v, u) = (0, 0)$. We have $\tilde{L}(v, u) = L(v) + F(u) = 0 \implies L(v) = -u$. But $L(v) \in \mathrm{Im}(L)$ and $-u \in U$. As $\mathrm{Im}(L) \cap U = \{0\}$, we get $L(v) = 0$ and $u = 0$. As L is injective, $L(v) = 0 \implies v = 0$.

To show that $\tilde{L}$ is surjective, let $w \in W$. As $W = U \oplus \mathrm{Im}(L)$, we can rewrite $w = u + L(v)$ for some $v \in V$. Hence, $\tilde{L}(u, v) = F(u) + L(v) = u + L(v) = w$.

Similarly, suppose that L is surjective and define $Z = \ker(L)$. Let $Y \subset V$ be a complement subspace of Z, i.e. $V = Y \oplus Z$. Define $G : V = Y \oplus Z \longrightarrow Z$ as $G(y + z) = z$, where $y \in Y$ and $z \in Z$.

Let's show that, with the choice made above, L' is an isomorphism.

To show that L' is injective, let $v \in \ker(L')$ and rewrite $v = y + z$ with $y \in Y$ and $z \in Z = \ker(L)$. As $v \in \ker(L')$, we have $(0, 0) = L'(v) = L'(y + z) = (L(y + z), G(y + z)) = (L(y), z)$, which implies

that $L(y)=0$ and $z=0$. But $L(y)=0 \implies y=0$, because $Y \cap \ker(L)=\{0\}$. Hence, $v=0$.
To show that L' is surjective, pick $(w,z) \in W \times Z$. As L is surjective, $w=L(v)$ for some $v \in V$. If we write $v=y+z$, with $y \in Y$ and $x \in Z$, we find that $w=L(v)=L(y+x)=L(y)+L(x)=L(y)$. As $Z=\ker(L)$, we have $G(z)=z$ and $z \in Z \subset V$. Hence, $L'(y+z)=(L(y+z),G(y+z))=(L(v),G(z))=(w,z)$.

Problem 3.
$AB=BA$ if and only if $(AB)_{ij}=(BA)_{ij}$ for all $i,j=1,\ldots,n$.
$(AB)_{ij}=\sum_{k=1}^{n} A_{ik}B_{kj}$ and $(BA)_{ij}=\sum_{h=1}^{n} B_{ih}A_{hj}$.
As A is diagonal, we have $A_{ij}=0$ if $i \neq j$.
So $(AB)_{ij}=A_{ii}B_{ij}=\lambda_i B_{ij}$, whereas $(BA)_{ij}=B_{ij}A_{jj}=\lambda_j B_{ij}$.
As $\lambda_i B_{ij}=\lambda_j B_{ij}$ for all $i,j=1,\ldots,n$ but $\lambda_i \neq \lambda_j$ for $i \neq j$, we obtain that $B_{ij}=0$ for $i \neq j$, which is the same as saying that B is diagonal.
Notice that, conversely, if B is diagonal, then $AB=BA$.

If the λ's are not all distinct, the previous conclusion is false. As an example, let $A=I_n$ be the identity matrix $n \times n$. Then $AB=BA$ for all matrices B, so also for nondiagonal B's.

Problem 4.
As V has finite dimension, then $\mathrm{ev}: V \longrightarrow V^{**}$ is a canonical isomorphism and we can use it to identify V with V^{**}.

We look for $\mathcal{B}=\{v_1,v_2,v_3\}$, so that $\mathrm{ev}_0=v_1^*$, $\mathrm{ev}_1=v_2^*$ and $\mathrm{ev}_i=v_3^*$. Hence, v_1 is a polynomial in $\mathbb{C}[t]_{\leq 2}$ such that $\mathrm{ev}_0(v_1)=1$, $\mathrm{ev}_1(v_1)=0$ and $\mathrm{ev}_i(v_1)=0$.
Thus, $v_1=-i(t-1)(t-i)=1+(i-1)t-it^2$,
v_2 is a polynomial in $\mathbb{C}[t]_{\leq 2}$ such that $\mathrm{ev}_0(v_2)=0$, $\mathrm{ev}_1(v_2)=1$ and $\mathrm{ev}_i(v_2)=0$.
Thus, $v_2=\frac{1+i}{2}t(t-i)=\frac{1+i}{2}t^2+\frac{1-i}{2}t$.
v_3 is a polynomial in $\mathbb{C}[t]_{\leq 2}$ such that $\mathrm{ev}_0(v_3)=0$, $\mathrm{ev}_1(v_3)=0$ and $\mathrm{ev}_i(v_3)=1$.
Thus, $v_3=-\frac{1+i}{2}t(t-1)=-\frac{1+i}{2}t^2+\frac{1+i}{2}t$,
Hence, $\mathcal{B}=\left\{1+(i-1)t-it^2, \frac{1+i}{2}t^2+\frac{1-i}{2}t, -\frac{1+i}{2}t^2+\frac{1+i}{2}t\right\}$.

To simplify the computation of $M_{\mathcal{B}^*}^{\mathcal{B}^*}(L^*)$, we write it as $M_{\mathcal{B}^*}^{\mathcal{B}^*}(L^*)=M_{\mathcal{B}^*}^{\mathcal{C}^*}\cdot M_{\mathcal{C}^*}^{\mathcal{B}^*}(L^*)$, where $\mathcal{C}=\{w_0=1,w_1=t,w_2=t^2\}$ and $\mathcal{C}^*=\{w_0^*,w_1^*,w_2^*\}$ is the dual basis.
We already know that $w_i^*(a_0+a_1t+a_2t^2)=a_i$.
Moreover, $M_{\mathcal{B}^*}^{\mathcal{C}^*}$ is the transpose of $M_{\mathcal{C}}^{\mathcal{B}}$.

Looking at v_1,v_2,v_3, we immediately see that $M_{\mathcal{C}}^{\mathcal{B}}=\frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 2i-2 & 1-i & 1+i \\ -2i & 1+i & -1-i \end{pmatrix}$,

so that $M_{\mathcal{B}^*}^{\mathcal{C}^*}=\frac{1}{2}\begin{pmatrix} 2 & 2i-2 & -2i \\ 0 & 1-i & 1+i \\ 0 & 1+i & -1-i \end{pmatrix}$.

Notice that $L^*(\mathrm{ev}_0)(a_0+a_1t+a_2t^2)=\mathrm{ev}_0\circ L(a_0+a_1t+a_2t^2)=\mathrm{ev}_0(a_1t+2a_2t^2)=0$.
Instead, $L^*(\mathrm{ev}_1)(a_0+a_1t+a_2t^2)=\mathrm{ev}_1\circ L(a_0+a_1t+a_2t^2)=\mathrm{ev}_1(a_1t+2a_2t^2)=a_1$, so that $L^*(\mathrm{ev}_1)=w_1^*$.
Similarly, $L^*(\mathrm{ev}_i)(a_0+a_1t+a_2t^2)=\mathrm{ev}_i\circ L(a_0+a_1t+a_2t^2)=\mathrm{ev}_i(a_1t+2a_2t^2)=-2a_2+a_1i$, so

that $L^*(\mathrm{ev}_i) = iw_1^* - 2w_2^*$.

For $M_{\mathcal{C}^*}^{\mathcal{B}^*}(L^*)$, its first column is given by $M_{\mathcal{C}^*}^{\mathcal{B}^*}(L^*)(e_1) = M_{\mathcal{C}^*} \circ L^* \circ M^{\mathcal{B}^*}(e_1) = M_{\mathcal{C}^*} \circ L(\mathrm{ev}_0) = 0$. The second column is given by $M_{\mathcal{C}^*}^{\mathcal{B}^*}(L^*)(e_2) = M_{\mathcal{C}^*} \circ L(\mathrm{ev}_1) = M_{\mathcal{C}^*}(w_1^*) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

The third column is given by $M_{\mathcal{C}^*}^{\mathcal{B}^*}(L^*)(e_3) = M_{\mathcal{C}^*} \circ L(\mathrm{ev}_i) = M_{\mathcal{C}^*}(iw_1^* - 2w_2^*) = \begin{pmatrix} 0 \\ i \\ -2 \end{pmatrix}$.

Hence, $M_{\mathcal{C}^*}^{\mathcal{B}^*}(L^*) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & i \\ 0 & 0 & -2 \end{pmatrix}$.

Finally, $M_{\mathcal{B}^*}^{\mathcal{B}^*} = M_{\mathcal{B}^*}^{\mathcal{C}^*} \cdot M_{\mathcal{C}^*}^{\mathcal{B}^*}(L^*) = \frac{1}{2}\begin{pmatrix} 2 & 2i-2 & -2i \\ 0 & 1-i & 1+i \\ 0 & 1+i & 1-i \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & i \\ 0 & 0 & -2 \end{pmatrix} =$

$$= \frac{1}{2}\begin{pmatrix} 0 & 2i-2 & -2+2i \\ 0 & 1-i & -1-i \\ 0 & 1+i & 1+3i \end{pmatrix}.$$