18.700 - Fall 2006 - Practice Final (180 minutes)

Problem 1.
Let $V = \mathbb{R}[t]_{\leq 3}$ and $T : V \longrightarrow V$ be the homomorphism defined as $T(p) = tp'' - p$.

(a) Compute the matrix $M_{\mathcal{B}}^{\mathcal{B}}(T)$, where $\mathcal{B} = \{1, t, t^2, t^3\}$, and the characteristic polynomial $p_f(t)$.

(b) Find the eigenvalues of f and its minimal polynomial.

(c) Find a basis $\mathcal{C}$ of V such that $M := M_{\mathcal{C}}^{\mathcal{C}}(T)$ is in Jordan form.

(d) What is the Jordan form of M^2 ?

Problem 2.
Let A be an $n \times n$ complex lower-triangular matrix such that $A_{1,1} = A_{2,2} = \cdots = A_{n,n} = \alpha \in \mathbb{C}$ and the product $A_{2,1}A_{3,2}A_{4,3}\cdots A_{n,n-1} \neq 0$. What is the Jordan form of A?

Problem 3.
Let $V = \mathcal{M}_{n\times n}(\mathbb{R})$ and $b : V \times V \longrightarrow \mathbb{R}$ the scalar product defined as $b(X, Y) = \mathrm{tr}(XY)$.
Given a matrix $A \in \mathcal{M}_{n\times n}(\mathbb{R})$, define $\varphi_A : V \longrightarrow V$ as $\varphi_A(X) = AX$.
Determine the matrices A such that φ_A is self-adjoint with respect to b.

Problem 4.
Let V be a real vector space of dimension n.

(a) Let $v \in V$ and define a scalar product on V^* as $b(f, g) = f(v)g(v)$ for $f, g \in V^*$. Determine $\mathrm{Rad}(b)$, the positivity and the negativity of b.

(b) Let $v_1, v_2 \in V$ and define a scalar product on V^* as $B(f, g) = f(v_1)g(v_1) + f(v_2)g(v_2)$ for $f, g \in V^*$. Determine $\mathrm{Rad}(B)$, the positivity and the negativity of B.

Problem 5.
Let $v, w \in \mathbb{R}^7$ and define $A = v \cdot {}^t w : \mathbb{R}^7 \longrightarrow \mathbb{R}^7$.
Determine characteristic and minimal polynomial of A and study the diagonalizability of A.

Problem 6.
Let $A \in \mathcal{M}_{3\times 3}(\mathbb{R})$ be a skew-symmetric real matrix of size 3.
Prove that: either $A = 0$ or there exists an $N \in \mathrm{GL}(3, \mathbb{R})$ such that $NA\,{}^t N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$.
Can you always choose N in $\mathrm{O}(3, \mathbb{R})$?