18.700 - Fall 2006 - Practice Final (180 minutes)

Problem 1.

(a) $T(1) = -1$, $T(t) = -t$, $T(t^2) = 2t - t^2$, $T(t^3) = 6t^2 - t^3$, so that $M^{\mathcal{B}}_{\mathcal{B}}(T) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & -1 \end{pmatrix}$.

The characteristic polynomial if $p_f(t) = (t+1)^4$.

(b) The unique eigenvalue of f is -1.
$\dim\ker(f + Id) = 2$ and $\ker(f + Id)$ is spanned by $\{1, t\}$.
$\dim\ker(f + Id)^2 = 3$ and $\ker(f + Id)^2$ is spanned by $\{1, t, t^2\}$.
$\dim\ker(f + Id)^3 = 4$ and $\ker(f + Id)^3 = V$.
Hence, the minimal polynomial is $p_{f,min}(t) = (t+1)^3$.

(c) Take $w_1 = t^3 \in V \setminus \ker(f + Id)^2$. Then define $w_2 = (f + Id)(w_1) = 6t^2 - t^3 + t^3 = 6t^2$ and $w_3 = (f + Id)^2(w_1) = (f + Id)(w_2) = (f + Id)(6t^2) = 6(2t - t^2 + t^2) = 12t$.
Moreover, let $u \in \ker(f + Id)$ such that $\{u, w_3\}$ is a basis of $\ker(f + Id)$. For example, $u = 1$.
Define $\mathcal{C} = \{w_3, w_2, w_1, u\} = \{t^3, 6t^2, 12t, 1\}$.

$$M := M^{\mathcal{C}}_{\mathcal{C}}(T) = \left(\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ \hline 0 & 0 & 0 & -1 \end{array}\right).$$

(d) $M^2 = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$ and $p_{M^2} = (t-1)^4$.

Moreover, $\dim\ker(M^2 - I) = 2$, $\dim\ker(M^2 - I)^2 = 3$ and so the Jordan form of M^2 is $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$.

Problem 2.

Let $V_i = \operatorname{span}\{e_i, e_{i+1}, \dots, e_n\}$ for $i = 1, \dots, n$ and define $V_{n+1} = \{0\}$.
Notice that $(A - \alpha I)(V_i) \subseteq V_{i+1}$ for $i = 1, \dots, n$.
$(A - \alpha I)(e_1) = A_{2,1}e_2 + v_3$ where $v_3 \in V_3$.
$(A - \alpha I)^2(e_1) = (A - \alpha I)(A_{2,1}e_2 + v_3) = A_{2,1}A_{3,2}e_3 + v_4$ with $v_4 \in V_4$.
Similarly, $(A - \alpha I)^{n-1}(e_1) = A_{2,1}A_{3,2} \cdots A_{n,n-1}e_n \neq 0$.
Hence, $(A - \alpha I)^{n-1} \neq 0$.
However, $(A - \alpha I)^n = 0$, which implies that the minimal polynomial of A is $p_{A,min}(t) = (t - \alpha)^n$.
This means that there is one Jordan block relative to the eigenvalue α of size n and so the Jordan form of A is $\begin{pmatrix} \alpha & 0 & 0 & \dots & 0 \\ 1 & \alpha & 0 & \dots & 0 \\ 0 & 1 & \alpha & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & \alpha \end{pmatrix}$.

Problem 3.
φ_A is self-adjoint for b if $b(\varphi_A(X), Y) = b(X, \varphi_A(Y))$ for every $X, Y \in V$, that is $\mathrm{tr}(AXY) = \mathrm{tr}(XAY)$ (or $\mathrm{tr}(YAX) = \mathrm{tr}(XAY)$, which is the same) for every $X, Y \in V$.
Call E^{ij} the matrix that has all zeroes except a 1 at the entry (i, j).
If $X = E^{ij}$ and $Y = E^{ji}$, then $\mathrm{tr}(XAY) = \mathrm{tr}(E^{ij}AE^{ji}) = A_{jj}$ and $\mathrm{tr}(YAX) = \mathrm{tr}(E^{ji}AE^{ij}) = A_{ii}$, so that we need $A_{ii} = A_{jj}$ for every $i, j = 1, \dots, n$.
If $X = E^{ij}$ and $Y = E^{jk}$ (with $k \neq i$), then $\mathrm{tr}(XAY) = \mathrm{tr}(E^{ij}AE^{jk}) = 0$ and $\mathrm{tr}(YAX) = \mathrm{tr}(E^{jk}AE^{ij}) = A_{ki}$, so that $A_{ki} = 0$ for $k \neq i$.
This implies that, if φ_A is self-adjoint for b, then A must be $A = \lambda I$ for some $\lambda \in \mathbb{F}$.
On the other hand, if $A = \lambda I$, it is clear that φ_A is self-adjoint because $\varphi_A(X) = \lambda X$ and so $b(\varphi_A(X), Y) = \lambda b(X, Y) = b(X, \varphi_A(Y))$.

Problem 4.

(a) If $v = 0$, then $b = 0$. So we assume $v \neq 0$.
Clearly, b is semi-positive definite, because $b(f, f) = f(v)^2 \geq 0$.
We can easily see that $f \in \mathrm{Rad}(b)$ if and only if $f(v) = 0$. In fact, if $f(v) = 0$, then $f \in \mathrm{Rad}(b)$. Instead, if $f(v) \neq 0$, then $b(f, f) = f(v)^2 \neq 0$ and so $f \notin \mathrm{Rad}(b)$.
Hence, $\mathrm{Rad}(b) = \mathrm{Ann}(v) = \{f \in V^* \mid f(v) = 0\}$, which has dimension $n - 1$.
This shows that the nullity of b ($= \dim \mathrm{Rad}(b)$) is $n - 1$ and the positivity of b is 1.

(b) If $v_1 = 0$ or $v_2 = 0$, we fall in case (a). So we assume $v_1 \neq 0$ and $v_2 \neq 0$.
As before, B is semi-positive definite, because $B(f, f) = f(v_1)^2 + f(v_2)^2 \geq 0$, so there is no negativity.
As before, $f \in \mathrm{Rad}(b)$ if and only if $f(v_1) = f(v_2) = 0$.
In fact, if $f(v_1) = f(v_2) = 0$, then $b(f, \cdot) = 0$ and so $f \in \mathrm{Rad}(b)$. On the other hand, if $f(v_1) \neq 0$ or $f(v_2) \neq 0$, then $b(f, f) = f(v_1)^2 + f(v_2)^2 > 0$ and so $f \notin \mathrm{Rad}(b)$.
Hence, $\mathrm{Rad}(b) = \mathrm{Ann}(\mathrm{span}\{v_1, v_2\})$ and we distinguish two cases. If $\{v_1, v_2\}$ are linearly independent, then $\dim \mathrm{Rad}(b) = n - 2$ and so the nullity is $n - 2$ and the positivity is 2.
If $\{v_1, v_2\}$ are linearly dependent, then $\dim \mathrm{Rad}(b) = n - 1$ and so the nullity is $n - 1$ and the positivity is 1.

Problem 5.
All the rows of $A = v \cdot {}^t w$ are proportional to each other, which means that $\mathrm{rk}(A) = 1$ and so $\dim \ker(A) = 6$.
The characteristic polynomial is $p_A(t) = t^6(\alpha - t)$ and one see immediately that $\alpha = \mathrm{tr}(A) = \langle v, w \rangle$.
If either v or w is zero, then $A = 0$. Let's assume $v, w \neq 0$, so that $A \neq 0$.
A is diagonalizable if and only if $\mathrm{tr}(A) = \langle v, w \rangle \neq 0$ and, in this case, $p_{A,min} = t(t - \langle v, w \rangle)$.
If v, w are orthogonal to each other, then $p_A(t) = -t^7$, $p_{A,min}(t) = t^2$ and A is not diagonalizable.

Problem 6.
If $0 \neq v \in \mathbb{R}^3$ is an eigenvector for A, then $Av = \lambda v$ and ${}^t vAv = \lambda\, {}^t vv = \lambda |v|^2$. On the other hand, ${}^t vAv = {}^t(-Av)v = -\lambda |v|^2$.
As $|v|^2 \neq 0$, then $\lambda = 0$, so that the only possible eigenvalue is 0.
If A is diagonalizable, then $A = 0$. So assume A not diagonalizable.
The characteristic polynomial $p_A(t)$ has degree 3 and so has a real root (i.e. A has an eigenvalue),

which thus must be zero.
Let $v \in \ker(A)$ be a vector of $|v|^2 = 1$ and let $W = \ker(A)^\perp$, so that $V = \ker(A) \oplus W$.
As $\ker(A)$ is A-invariant and A is skew-symmetric (and so skew-self-adjoint for the standard scalar product on $\mathbb{R}^3$), also W is A-invariant. Let $\{w_1, w_2\}$ be any orthonormal basis of W, so that $\{v, w_1, w_2\}$ is an orthonormal basis of R^3.
Let M be the orthogonal matrix that takes e_1, e_2, e_3 to v, w_1, w_2.
We get ${}^tMAM = M^{-1}AM = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -a \\ 0 & a & 0 \end{pmatrix}$, with $a \in \mathbb{R}$. If $a > 0$, then let $u_1 = w_1/\sqrt{a}$ and $u_2 = w_2/\sqrt{a}$. If $a < 0$, then let $u_1 = w_2/\sqrt{a}$ and $u_2 = w_1/\sqrt{a}$. Let N be the invertible matrix that takes e_1, e_2, e_3 to v, u_1, u_2.
We get ${}^tNAN = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$. Call this matrix P.

We cannot always choose $N \in \mathrm{O}(3, \mathbb{R})$.
By contradiction, if we could write ${}^tNAN = N^{-1}AN = P$ with $N \in \mathrm{O}(3, \mathbb{R})$, then ${}^tN = N^{-1}$ and so $\mathrm{tr}(A^2) = \mathrm{tr}(N^{-1}A^2N) = \mathrm{tr}(P^2) = -2$, but if $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -a \\ 0 & a & 0 \end{pmatrix}$, then $A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -a^2 & 0 \\ 0 & 0 & -a^2 \end{pmatrix}$ and $\mathrm{tr}(A^2) = -2a^2$, which is $\neq 2$ if $a \neq \pm 1$.