18.700 - Fall 2006 - Practice Exam F

Problem 1.
Let $A : \mathbb{R}^5 \longrightarrow \mathbb{R}^5$ be the homomorphism of real vector spaces represented by

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 & 1 \\ 0 & -3 & -1 & 3 & -3 \\ 0 & -3 & -1 & 1 & -1 \end{pmatrix}$$

(a) Determine the characteristic and the minimal polynomial of A.
Is A triangularizable in $\mathcal{M}_{5\times 5}(\mathbb{R})$?

(b) Determine for which $k = 1, 2, 3, 4$ there is an A-invariant subspace $W_k \subset \mathbb{R}^5$ of dimension k.

(c) Consider A as a matrix in $\mathcal{M}_{5\times 5}(\mathbb{C})$. Is A diagonalizable over $\mathbb{C}$?

Problem 2.
Let $f : V \longrightarrow V$ be an endomorphism of a vector space V of dimension n over the field $\mathbb{F}$. Show that there exists an integer $k \leq n$ such that all the following conditions are simultaneously satisfied:

(a) $\ker(f^k) = \ker(f^{k+1})$

(b) $\mathrm{Im}(f^k) = \mathrm{Im}(f^{k+1})$ and $f|_{\mathrm{Im}(f^k)}^{\mathrm{Im}(f^k)} : \mathrm{Im}(f^k) \longrightarrow \mathrm{Im}(f^k)$ is an isomorphism

(c) $f(\ker(f^k)) \subseteq \ker(f^k)$ and $f|_{\ker(f^k)}^{\ker(f^k)} : \ker(f^k) \longrightarrow \ker(f^k)$ is nilpotent[1]

(d) $V = \ker(f^k) \oplus \mathrm{Im}(f^k)$.

Problem 3.
Let $A, B \in \mathcal{M}_{n\times n}(\mathbb{F})$ be diagonalizable matrices. We say that A and B are *simultaneously diagonalizable* if there exists an invertible $M \in \mathcal{M}_{n\times n}(\mathbb{F})$ such that MAM^{-1} and MBM^{-1} are both diagonal.
Prove that A and B are simultaneously diagonalizable if and only if $AB = BA$.

Problem 4.
Let $A \in \mathcal{M}_{m\times n}(\mathbb{R})$ be a matrix with all entries in $\mathbb{Z}$. Show that $\ker(A)$ has a basis of vectors whose components are in $\mathbb{Z}$.

[1] An endomorphism $g : W \longrightarrow W$ is **nilpotent** if $g^m = 0$ for some positive integer m.