18.700 - Fall 2006 - Solutions to Practice Exam F

Problem 1.

(a) By direct computation, $p_A(t) = (2-t)^3(3+t^2)$.
Moreover, $\ker(A-2I)$ has dimension 2, so that $\ker(A-2I)^2$ must have dimension 3. Hence, the minimal polynomial if $p_{A,min}(t) = (2-t)^2(3+t^2)$.
A is not triangularizable, because $(3+t^2)$ is irreducible over $\mathbb{R}$ (it has no roots in $\mathbb{R}$) and $(3+t^2)$ has degree 2.

(b) Let $\{v_1, v_2\}$ be a basis of $\ker(A-2I)$ and let $\{u_1, u_2\}$ be a basis of $\ker(A^2+3I)$. Let $\{v_1, v_2, v_3\}$ be a basis of $\ker(A-2I)^2$.
For instance, we can define $W_1 = \mathrm{span}\{v_1\}$, $W_2 = \mathrm{span}\{v_1, v_2\}$, $W_3 = \mathrm{span}\{u_1, u_2, v_1\}$, $W_4 = \mathrm{span}\{u_1, u_2, v_1, v_2\}$ (but there are many possibilities).

(c) No, it is not. In fact, A is diagonalizable if and only if $\mu_{alg}(e) = \mu_{geom}(e)$ for every $e \in \mathbb{F}$. However, we already discovered that $\mu_{alg}(2) = 3 > 2 = \mu_{geom}(2)$. Hence, A is not diagonalizable.

Problem 2.

(a) Notice that $\{0\} \subseteq \ker(f) \subseteq \ker(f^2) \subseteq \ker(f^3) \subseteq \ldots$ and the inclusions cannot be all strict, because V has finite dimension. Hence, there exists $k \leq n$ such that $\ker(f^k) = \ker(f^{k+1})$.

(b) Notice that $\mathrm{Im}(f^i) \supseteq \mathrm{Im}(f^{i+1})$ and $\dim \mathrm{Im}(f^i) + \dim\ker(f^i) = n$, so $\mathrm{Im}(f^k) = \mathrm{Im}(f^{k+1})$.
Moreover, $f|_{\mathrm{Im}(f^i)}^{\mathrm{Im}(f^{i+1})} : \mathrm{Im}(f^i) \longrightarrow \mathrm{Im}(f^{i+1}) \subseteq \mathrm{Im}(f^i)$ is surjective. Hence, $f|_{\mathrm{Im}(f^k)}^{\mathrm{Im}(f^k)} : \mathrm{Im}(f^k) \longrightarrow \mathrm{Im}(f^k)$ is an isomorphism (by dimensional reasons).

(c) Clearly, $f(\ker(f^k)) \subseteq \ker(f^{k-1}) \subseteq \ker(f^k)$.
In fact, if $v \in \ker(f^k)$, then $f^{k-1}(f(v)) = f^k(v) = 0$.
Moreover, $\left(f|_{\ker(f^k)}^{\ker(f^k)}\right)^k = (f^k)|_{\ker(f^k)}^{\ker(f^k)} = 0$, so that $f|_{\ker(f^k)}^{\ker(f^k)}$ is nilpotent.

(d) Because, the restriction of f to $\mathrm{Im}(f^k)$ is an isomorphism, then also the restriction of f^k to $\mathrm{Im}(f^k)$ is an isomorphism, and so in particular is injective. This implies that $\mathrm{Im}(f^k) \cap \ker(f^k) = \{0\}$.
So, $\ker(f^k)$ and $\mathrm{Im}(f^k)$ are subspaces of V that intersect only in $\{0\}$ and such that $\dim\ker(f^k) + \dim \mathrm{Im}(f^k) = \dim V$. Hence, $V = \ker(f^k) \oplus \mathrm{Im}(f^k)$.

Problem 3.
If A, B are simultaneously diagonalizable, then $\exists M$ invertible such that $MAM^{-1} = D_1$ and $MBM^{-1} = D_2$, with D_1 and D_2 diagonal. Because $D_1D_2 = D_2D_1$, we obtain $AB = (M^{-1}D_1M)(M^{-1}D_2M) = M^{-1}D_1D_2M = M^{-1}D_2D_1M = (M^{-1}D_2M)(M^{-1}D_1M) = BA$.

Suppose conversely that $AB = BA$. We want to find a basis $\mathcal{B} = \{u_1, \ldots, u_n\}$ of $\mathbb{F}^n$ such that u_i is an eigenvector for A and B for every $i = 1, \ldots, n$. This would imply that $M_{\mathcal{B}}^{\mathcal{B}}(A)$ and $M_{\mathcal{B}}^{\mathcal{B}}(B)$ are diagonal, and so A and B are simultaneously diagonalizable.
Let $e_1, \ldots, e_k$ be the eigenvalues of A and $\mathbb{F}^n = E_{e_1,A} \oplus \ldots E_{e_k,A}$, where $E_{e_i,A}$ is the eigenspace

$\ker(A - e_i I)$.
For every $v \in E_{e_i,A}$ we have $A(Bv) = B(Av) = B(e_i v) = e_i(Bv)$, so that $Bv \in E_{e_i,A}$. Hence, $B(E_{e_i,A}) \subseteq E_{e_i,A}$.
Let $0 \neq w \in V$ be an eigenvector for B of eigenvalue μ. Then $w = v_1 + \cdots + v_k$ for unique vectors $v_i \in E_{e_i,A}$ ($w \neq 0$ implies that not all the v_i's are zero) and also $\mu w = \mu v_1 + \cdots + \mu v_k$.
Applying B on both sides of $w = v_1 + \cdots + v_k$, we get $\mu w = Bw = Bv_1 + \cdots + Bv_k$.
As $\mathbb{F}^n = E_{e_1,A} \oplus \dots E_{e_k,A}$, we have $Bv_i = \mu v_i$ for all $i = 1, \dots, k$, and so the v_i's are common eigenvectors for A and B. As $w \neq 0$, there is at least one v_i which is not zero, so that we find at least one common eigenvector.
Repeating this procedure for $w_1, \dots, w_n$, where $\{w_1, \dots, w_n\}$ is a basis of eigenvectors of B, we obtain a set $\mathcal{C}$ of at least n common eigenvectors. These eigenvectors in $\mathcal{C}$ generate $\mathbb{F}^n$, because they generate $\{w_1, \dots, w_n\}$. Hence, we can extract from $\mathcal{C}$ a basis of $\mathbb{F}^n$, which is a basis of common eigenvectors for A and B.

Problem 4.

Clearly, it is sufficient to show that we can find a basis of $\ker(A)$ made of vectors with entries in $\mathbb{Q}$, because then we can multiply these vectors by a suitable integer and get vectors with entries in $\mathbb{Z}$.
As the matrix A has rational (even integral!) entries, the associated homomorphism $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ sends a vector of $\mathbb{Q}^n$ (which is contained in $\mathbb{R}^n$) to a vector of $\mathbb{Q}^n$. Call $\tilde{A} : \mathbb{Q}^n \longrightarrow \mathbb{Q}^n$ the restriction of A to $\mathbb{Q}^n$.
Clearly, $\tilde{A}$ is represented by the same matrix as A.
Pick a basis $\{v_1, \dots, v_k\}$ over $\mathbb{Q}$ of $\ker(\tilde{A})$. Then $v_i \in \mathbb{Q}^n \subset \mathbb{R}^n$. Hence, $\{v_1, \dots, v_k\}$ is also a basis over $\mathbb{R}$ of $\ker(A)$ and each v_i has rational entries.