18.700 - Fall 2006 - Some other random exercises - E

Problem 1.

Let $M \in \mathcal{M}_{n\times n}(\mathbb{R})$ be an upper triangular matrix such that $M_{ii} = 1$ for all $i = 1, 2, \ldots, n$. Suppose that there exists an integer $k \geq 1$ such that $M^k = I$. Prove that $M = I$.

Problem 2.

Let $N \in \mathcal{M}_{n\times n}(\mathbb{R})$ be a strictly upper triangular matrix such that $N_{i,i+1} \neq 0$ for all $i = 1, \ldots, n-1$. Prove that $N^k \neq 0$ for every $k = 1, \ldots, n-1$. Conclude that N is similar to A, where $A_{i+1,i} = 1$ and $A_{i,j} = 0$ if $i \neq j+1$.

Problem 3.

Show that the minimal polynomial $p_{min,M}$ of the matrix $M = \left(\begin{array}{c|c} A_{n\times n} & 0_{n\times m} \\ \hline 0_{m\times n} & B_{m\times m} \end{array}\right)$ is the least common multiple of $p_{min,A}$ and $p_{min,B}$.

Problem 4.

Let $f : V \longrightarrow V$ a homomorphism of $\mathbb{F}$-vector spaces, where $\dim_{\mathbb{F}} V = n$.

(a) Suppose that $f^2 = f$. Show that there exists a basis $\mathcal{B}$ of V such that

$$M_{\mathcal{B}}^{\mathcal{B}}(f) = \left(\begin{array}{c|c} I_{k\times k} & 0 \\ \hline 0 & 0_{(n-k)\times(n-k)} \end{array}\right)$$

with k uniquely determined.

(b) Suppose that $f^2 = I$. Show that there exists a basis $\mathcal{B}$ of V such that

$$M_{\mathcal{B}}^{\mathcal{B}}(f) = \left(\begin{array}{c|c} I_{k\times k} & 0 \\ \hline 0 & -I_{(n-k)\times(n-k)} \end{array}\right)$$

with k uniquely determined.

(c) Suppose that $f^2 = -I$ and $\mathbb{F} = \mathbb{R}$. Show that there exists a basis $\mathcal{B}$ of V such that

$$M_{\mathcal{B}}^{\mathcal{B}}(f) = \left(\begin{array}{c|c} 0 & -I_{k\times k} \\ \hline I_{k\times k} & 0 \end{array}\right)$$

so that $n = 2k$ is even.

Problem 5.

Let $A, B \in \mathcal{M}_{n\times n}(\mathbb{F})$ be fixed matrices and consider the set $E(A, B) = \{X \in \mathcal{M}_{n\times n}(\mathbb{F}) \mid AX = XB\}$. Prove that:

(a) $E(A, B)$ is a vector subspace of $\mathcal{M}_{n\times n}(\mathbb{F})$

(b) If A' is similar to A and B' is similar to B, then $E(A', B')$ is isomorphic to $E(A, B)$.

Problem 6.

Let V be a vector space over $\mathbb{F}$ and $f : V \longrightarrow V$ be a homomorphism. Let $W_1, W_2 \subseteq V$ be f-invariant vector subspaces such that $V = W_1 + W_2$ and suppose that $f\big|_{W_1}^{W_1} : W_1 \longrightarrow W_1$ and $f\big|_{W_2}^{W_2} : W_2 \longrightarrow W_2$ are triangulable over $\mathbb{F}$. Is f necessarily triangulable over $\mathbb{F}$?

Problem 7.

Let V be a vector space of dimension n over $\mathbb{F}$ and let $L : V \longrightarrow V$ be an endomorphism and k be an integer such that $0 < k < n$. Prove that all subspaces of V of dimension k are L-invariant if and only if $L = cI$ for some $c \in \mathbb{F}$.

Problem 8.

Let $A, B \in \mathcal{M}_{n\times n}(\mathbb{R}) \subset \mathcal{M}_{n\times n}(\mathbb{C})$. Show that A, B are similar in $\mathcal{M}_{n\times n}(\mathbb{R})$ if and only if they are similar in $\mathcal{M}_{n\times n}(\mathbb{C})$.

Problem 9.

Show that $\mathcal{D} = \{A \in \mathcal{M}_{n\times n}(\mathbb{R}) \,|\, A \text{ is diagonalizable}\} \subset \mathcal{M}_{n\times n}(\mathbb{R})$ generates the whole $\mathcal{M}_{n\times n}(\mathbb{R})$ as a vector space.
Show that the same holds if we replace $\mathbb{R}$ by $\mathbb{Q}$ or $\mathbb{C}$.