18.700 - Fall 2006 - Solutions to Practice Exercises E

Problem 1.

Clearly, as M is upper triangular with $M_{ii} = 1$ for all i, the characteristic polynomial of M is $p_M(\lambda) = (1-\lambda)^n$.
Hence, the minimal polynomial of M is $p_{M,min} = (\lambda - 1)^k$ with $1 \leq k \leq n$ (because $p_{M,min}$ divides p_M).
Moreover, $M^k = I$ implies that the polynomial $\lambda^k - 1$ belongs to $I_M = (p_{M,min})$, so that $p_{M,min}$ divides $\lambda^k - 1 = (\lambda - 1)(\lambda^{k-1} + \lambda^{k-2} + \cdots + \lambda + 1)$.
Because $\lambda^{k-1} + \cdots + \lambda + 1$ is not divisible by $\lambda - 1$ (remember that $\mathbb{F} = \mathbb{R}$!), then $p_{M,min}$ divides $\lambda - 1$, and so $p_{M,min} = \lambda - 1$. This means that $0 = p_{M,min}(M) = M - I$, that is $M = I$.

Problem 2.

As we have seen in the past, if we define $V_k = \text{span}\{e_1, \ldots, e_i\}$, then $\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq V_n = \mathbb{R}^n$ is an N-invariant flag and $N^k(V_i) \subseteq V_{i-k}$. Moreover, the hypothesis tells us that $N(V_i) = V_{i-1}$, so that $N^k(V_i) = V_{i-k}$. Hence, $\ker(N^k) = V_k$, which implies that $N^k \neq 0$ for $k < n$.
Moreover, take the basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ defined as $v_i = N^{n-i} e_n$. It is indeed as basis because $v_i \in V_i \setminus V_{i-1}$.
You can check that $M_{\mathcal{B}}^{\mathcal{B}}(N) = A$.

Problem 3.

Let q be the monic polynomial which is the least common multiple of $p_{min,A}$ and $p_{min,B}$.
In particular, both $p_{min,A}$ and $p_{min,B}$ divide q.
As $r(M) = \left(\begin{array}{c|c} r(A) & 0 \\ \hline 0 & r(B) \end{array}\right)$ for every $r \in \mathbb{F}[t]$, then $q(M) = 0$ and so the minimal polynomial of M divides q.
Moreover, as $p_{M,min}(M) = 0$, we must have $p_{M,min}(A) = 0$ and $p_{M,min}(B) = 0$, so that both $p_{A,min}$ and $p_{B,min}$ divide $p_{M,min}$. As a consequence, q divides $p_{M,min}$ and so they differ by a multiplicative (nonzero) factor. As q and $p_{M,min}$ are both monic, $q = p_{M,min}$.

Problem 4.

(a) Call $E_0 = \ker(f)$ and $E_1 = \ker(f - I)$.
$f^2 = f$ implies that $t(t-1)$ belongs to $I_f \subset \mathbb{F}[t]$.
As $t, t-1 \neq 0$ and $(t, t-1) = 1$, there exist $\alpha(t), \beta(t) \in \mathbb{F}[t]$ such that $\alpha(t)t + \beta(t)(t-1) = 1$.
Evaluating over f gives $\alpha(f)f + \beta(f)(f - I) = I \in \text{End}(V)$.
Hence, for every $v \in V$, $f[\alpha(f)(v)] + (f - I)[\beta(f)(v)] = v$ and $f\alpha(f)(v) \in E_1$ and $(f - I)\beta(f)(v) \in E_0$.
This shows that $E_0 \oplus E_1 = V$.
If $\text{rk}(f) = k$, then $\dim E_1 = k$ and $\dim E_0 = n - k$.
Let $\{v_1, \ldots, v_k\}$ be a basis of E_1 and $\{v_{k+1}, \ldots, v_n\}$ be a basis of E_0.
$\mathcal{B} = \{v_1, \ldots, v_n\}$ is a basis of $V = E_0 \oplus E_1$ and finally $M_{\mathcal{B}}^{\mathcal{B}}(f)$ is of the wanted form.
Notice that k is uniquely determined because $k = \dim \text{Im}(f)$.

(b) The same argument as in (a) applies, noticing that $(t-1)(t+1) \in I_f$ and that $(t-1, t+1) = 1$.

In this case, $2k - n = \mathrm{tr}(f)$ and so k is uniquely determined.

(c) Notice that $\det(f)^2 = \det(f^2) = \det(-I) = (-1)^n$. As V is a real vector space, this implies that $n = 2k$ is even and that f is invertible.
Notice also that $t^2 + 1 \in I_f$ and $t^2 + 1$ is irreducible over $\mathbb{R}$, so that $p_{f,min}(t) = t^2 + 1$. In particular, f has no eigenvectors.
We want to choose the basis $\mathcal{B} = \{v_1, w_1, v_2, w_2, \dots, v_k, w_k\}$ in the following way.
Pick any $0 \neq v_1 \in V$ and let $w_1 = f(v_1)$. Notice that w_1 is not a multiple of v_1, because f has no eigenvectors. Hence, $\{v_1, w_1\}$ is a set of linearly independent vectors.
In order to proceed by induction, assume that $\{v_1, w_1, \dots, v_i, w_i\}$ is a set of linearly independent vectors for $i < k$, with $w_j = f(v_j)$. Pick $v_{i+1} \in V$ not in $V_i := \mathrm{span}\{v_1, w_1, \dots, v_i, w_i\}$ and let $w_{i+1} = f(v_{i+1})$. We want to show that $\{v_1, w_1, \dots, v_{i+1}, w_{i+1}\}$ is a set of linearly independent vectors.
As $\{v_1, w_1, \dots, v_{i+1}\}$ is a set of linearly independent vectors by construction, suppose by contradiction that $w_{i+1} = a_{i+1}v_{i+1} + u$, with $u \in V_i$.
Applying f we get $-v_{i+1} = a_{i+1}w_{i+1} + f(u) = a_{i+1}(a_{i+1}v_{i+1} + u) + f(u) = a_{i+1}^2 v_{i+1} + x$, with $x \in V_i$ (because V_i is f-invariant). So we obtain $(1 + a_{i+1}^2)v_{i+1} + x = 0$ with $x \in V_i$ and $1 + a_{i+1}^2 \neq 0$ because $a_j \in \mathbb{R}$, which is clearly a contradiction because $v_{i+1} \notin V_i$.
It is easy to see that $\mathcal{B}$ is the wished basis.

Problem 5.

Part (a) is just routine check.
If $A \sim A'$, then $A' = MAM^{-1}$. If $B \sim B'$, then $B' = NBN^{-1}$.
Hence, $E(A', B') = \{X \in \mathcal{M}_{n\times n}(\mathbb{F}) \,|\, MAM^{-1}X = XNBN^{-1}\}$.
Notice that $MAM^{-1}X = XNBN^{-1} \iff A(M^{-1}XN) = (M^{-1}XN)B$.
Define $g : E(A', B') \longrightarrow E(A, B)$ as $g(X) = M^{-1}XN$. This is clearly a homomorphism and it is invertible, because $g^{-1}(Y) = MYN^{-1}$.

Problem 6.

Yes.
Remember that f is triangulable if and only if the characteristic polynomial $p_f(t)$ is completely factorizable over $\mathbb{F}$. This condition is equivalent to having a completely factorizable minimal polynomial $p_{f,min}$ (the reason will be explained on Nov.14).
Call f_1 the restriction of f to W_1 and f_2 the restriction of f to W_2.
Let p_{f_i} be the characteristic polynomial of f_i. Then $p_{f_1} \cdot p_{f_2}$ belongs to I_f. In fact, for every $v \in V$ we can write $v = w_1 + w_2$ for some $w_1 \in W_1$ and $w_2 \in W_2$. Hence, $(p_{f_1}p_{f_2})(f)(v) = p_{f_2}(f)[p_{f_1}(f)(w_1)] + p_{f_1}(f)[p_{f_2}(f)(w_2)] = 0$ (by Cayley-Hamilton applied to W_1 and W_2).
Again by Cayley-Hamilton, this means that $p_{f,min}$ divides $p_{f_1}p_{f_2}$, which is completely factorizable, and so $p_{f,min}$ is.

Problem 7.

Let's prove first the case $k = 1$.
Let $0 \neq v \in V$. The condition implies that $L(v) = cv$ for some $c \in \mathbb{F}$. Notice that this c might well depend on the chosen $v \in V$ a priori. We want to show that it does not.
Suppose $L(v_1) = c_1v_1$ and $L(v_2) = c_2v_2$.

Then $c_1 v_1 + c_2 v_2 = L(v_1 + v_2) = c_3(v_1 + v_2)$, where the equality on the right is again due to the L-invariance of $\operatorname{span}\{v_1 + v_2\}$.
This implies that $c_1 = c_2$, so that the constant c does not depend on the vector and we have showed that there exists a $c \in \mathbb{F}$ such that $L(v) = cv$ for every $v \in V$.
If $k > 1$, we want to show that every 1-dimensional vector subspace $W \subset V$ is L-invariant and then conclude, using the result above.
This is easy, because every 1-dimensional W can be realized as the intersection of $k-1$ subspaces $Z_1, \dots, Z_{k-1} \subseteq V$ of dimension k. Finally, we conclude noticing that $f(W) = f(Z_1 \cap \dots \cap Z_{k-1}) \subseteq f(Z_1) \cap \dots \cap f(Z_{k-1}) \subseteq Z_1 \cap \dots \cap Z_{k-1} = W$, so that W is L-invariant.

Problem 8.
Clearly, if $A \sim B$ in $\mathcal{M}_{n\times n}(\mathbb{R})$, then they are similar in $\mathcal{M}_{n\times n}(\mathbb{C})$.
Vice versa, let $M \in \mathcal{M}_{n\times n}(\mathbb{C})$ such that $AM = MB$ and M is invertible.
Then we can write $M = X + iY$, where $X, Y \in \mathcal{M}_{n\times n}(\mathbb{R})$.
From $A(X + iY) = (X + iY)B$, matching real and imaginary parts, we obtain $AX = XB$ and $AY = YB$ (because A, B, X, Y are real matrices).
We would be done if either X or Y is invertible.
If they are not, consider the matrix $P_c = X + cY$ with $c \in \mathbb{C}$.
Clearly, $AP_c = P_c B$. The question becomes: can we find a value $c \in \mathbb{R}$ such that P_c is invertible?
Remember that P_c is invertible if and only if $\det(P_c) \neq 0$.
On the other hand, $q(c) := \det(P_c) \in \mathbb{R}[c]$ is a polynomial in the variable c (of degree at most n) and $q(c)$ is not the zero polynomial, because $P_i = M$ is invertible and so $q(i) \neq 0$.
Hence, there exist at most n values of $c \in \mathbb{R}$ (even in $\mathbb{C}$) such that $q(c) = 0$ and so P_c is not invertible.
Pick a $c \in \mathbb{R}$ such that $q(c) \neq 0$. Then P_c is invertible and $AP_c = P_c B$ gives that $A \sim B$ in $\mathcal{M}_{n\times n}(\mathbb{R})$.

Problem 9.
A proof the works for any infinite field $\mathbb{F}$.
Every matrix A can be written as $A = U + D + L$, where A is strictly upper triangular, D is diagonal and L is strictly lower triangular.
We need to show that U and L can be written as a linear combination of diagonalizable matrices. We will do only for U: the argument for L is identical.
Let $B \in \mathcal{M}_{n\times n}(\mathbb{F})$ be a diagonal matrix such that $B_{ii} \neq B_{jj}$ whenever $i \neq j$ (for example, $B_{ii} = i$ works if $\operatorname{char}(\mathbb{F}) = 0$ as for $\mathbb{F} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$).
Then the characteristic polynomials of $U + B$ and $U - B$ have distinct roots, so both $U + B$ and $U - B$ are diagonalizable and B can be rewritten as $B = \dfrac{U+B}{2} + \dfrac{U-B}{2}$.

A different proof that works on any field $\mathbb{F}$.
Let $E^{(ij)}$ be the $n \times n$ matrix (with coefficients in $\mathbb{F}$) whose entries are all zeroes, except a 1 at the position (i,j).
Clearly, $\{E^{(ij)} \mid 1 \leq i, j \leq n\}$ is a basis of $\mathcal{M}_{n\times n}(\mathbb{F})$ and $E^{(ii)}$ is diagonalizable for $i = 1, \dots, n$.
Moreover, $E^{(ij)} + E^{(jj)}$ is also diagonalizable, so that $E^{(ij)}$ is also contained in $\mathcal{D}$.