# 18.700 - Fall 2006 - Practice Exam C

**Problem 1.**
Let $\mathcal{M}_{n\times n}(\mathbb{F})$ be the $\mathbb{F}$-vector space of $n \times n$ matrices, where $\mathbb{F}$ is a fixed field.
Let $1 \leq k \leq n$ be an integer and define $R_k \subset \mathcal{M}_{n\times n}(\mathbb{F})$ to be the subset of matrices of rank $k$. [1]
Show that the span of $R_k$ is the whole $\mathcal{M}_{n\times n}(\mathbb{F})$.
*(Hint: for $R_1$ it is very easy; try with $R_2$ to understand how it works in general.)*

**Problem 2.**
Let $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the homomorphism of real vector spaces given by $L = \begin{pmatrix} 2 & -2 & 1 \\ 3 & -4 & 1 \end{pmatrix}$ and let
$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \right\}$ be a basis of $\mathbb{R}^3$ and $\mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ a basis of $\mathbb{R}^2$.
Write the matrix $M_{\mathcal{C}}^{\mathcal{B}}(L)$ associated to $L$ with respect to the bases $\mathcal{B}$ and $\mathcal{C}$.

**Problem 3.**
Let $V$ be a vector space of dimension $n$ over the field $\mathbb{F}$. Let $T : V \longrightarrow V$ be a fixed homomorphism.
Consider the subsets $\mathcal{L} = \{S \in \mathrm{Hom}(V,V) \,|\, ST = 0\}$ and $\mathcal{R} = \{S \in \mathrm{Hom}(V,V) \,|\, TS = 0\}$ of $\mathrm{Hom}(V,V)$.
Are $\mathcal{L}$ and $\mathcal{R}$ vector subspaces of $\mathrm{Hom}(V,V)$?
What is the dimension of $\mathrm{span}(\mathcal{L})$, $\mathrm{span}(\mathcal{R})$ and $\mathrm{span}(\mathcal{L} \cap \mathcal{R})$?
Does the subset $\mathcal{L} \cup \mathcal{R}$ span the whole $\mathrm{Hom}(V,V)$?

**Problem 4.**
Let $W = \mathrm{span}_{\mathbb{C}}\{v_1, v_2\} \subset \mathbb{C}^3$ be the vector subspace (over $\mathbb{C}$) spanned by $\{v_1, v_2\}$,
where $v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.
Call $j : W \longrightarrow \mathbb{C}^3$ the inclusion and let $j^* : (\mathbb{C}^3)^* \longrightarrow W^*$ be its dual homomorphism.
Determine the $\ker(j^*)$.
Consider the basis of $\mathbb{C}^3$ given by $\mathcal{B} = \{v_1, v_2, e_1\}$, where $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.
Determine the dual basis $\mathcal{B}^*$ of $(\mathbb{C}^3)^*$.

[1] Remember that the *rank* of a matrix $A$ is the dimension of the image of the homomorphism represented by $A$.