18.700 - Fall 2006 - Solutions of Practice Exam C

Problem 1.

Call M^{ij} the matrix whose unique nonzero entry is a 1, placed at the i-th row j-th column. Clearly, M^{ij} has rank 1 and clearly the set $\{M^{ij} \in \mathcal{M}_{n\times n}(\mathbb{F}) \,|\, i,j = 1,\ldots,n\}$ generate $\mathcal{M}_{n\times n}(\mathbb{F})$ (really, they are even a basis).

For $k > 1$, we will show that we can generate R_1 using matrices in R_k. As R_1 is a set of generators, it will follow that R_k is a set of generators as well.

For notational simplicity, let's use the following convention: consider the index for the row and for the column as a number in $\mathbb{Z}/n$, in such a way that the $\bar{0}$-th column is the n-th column, the $\overline{n+1}$-st column is the first column and so on.

Suppose I want to generate M^{ij} with matrices in R_k.

Define the matrix A as $A_{rc} = 1$ for $(r,c) = (i-1,j),(i,j),(i,j+1),(i+1,j+2),(i+2,j+3),\ldots,(i+k-1,j+k)$ and $A_{rc} = 0$ otherwise. Define also B as $B_{rc} = 1$ for $(r,c) = (i-1,j),(i,j+1),(i+1,j+2),\ldots,(i+k-1,j+k)$ and $B_{rc} = 0$ otherwise.

Both A and B have rank k and $A - B = M^{ij}$.

Problem 2.

Call $v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $v_2 = \begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$, $w_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $w_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$$L(v_1) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} = -3w_1 + 4w_2, \quad L(v_2) = \begin{pmatrix} -14 \\ -26 \end{pmatrix} = -12w_1 - 2w_2 \quad \text{and}$$

$$L(v_3) = \begin{pmatrix} -4 \\ -6 \end{pmatrix} = -2w_1 - 2w_2.$$

Hence, $M^{\mathcal{B}}_{\mathcal{C}}(L) = \begin{pmatrix} -3 & -12 & -2 \\ 4 & -2 & -2 \end{pmatrix}$

Problem 3.

$\mathcal{L}$ is a vector subspace, because: if $S_1, S_2 \in \mathcal{L}$, then $S_1T = S_2T = 0$ and so $(S_1+S_2)T = S_1T + S_2T = 0$. Similarly, $(\lambda S_1)T = \lambda(S_1T) = 0$ for $\lambda \in \mathbb{F}$ and $S_1 \in \mathcal{L}$.

The proof for $\mathcal{R}$ is analogous.

Hence, $\mathrm{span}(\mathcal{L}) = \mathcal{L}$, $\mathrm{span}(\mathcal{R}) = \mathcal{R}$, $\mathrm{span}(\mathcal{L}\cap\mathcal{R}) = \mathcal{L}\cap\mathcal{R}$ and $\mathrm{span}(\mathcal{L}\cup\mathcal{R}) = \mathcal{L}+\mathcal{R}$.

Let $k = \dim \mathrm{Im}(T)$, so that $\dim\ker(T) = n - k$.

Let $\{v_1,\ldots,v_k\}$ be a basis of $\mathrm{Im}(T)$ and $\{w_1,\ldots,w_{m-k}\}$ a basis of $\ker(T)$. Complete them to bases $\mathcal{B} = \{v_1,\ldots,v_n\}$ and $\mathcal{C} = \{w_1,\ldots,w_n\}$ of V.

We can thus define an isomorphism $\Phi : \mathrm{Hom}(V,V) \longrightarrow \mathcal{M}_{n\times n}(\mathbb{F})$ given by $\Phi(S) = M^{\mathcal{B}}_{\mathcal{C}}(S)$.

Through this isomorphism, it is clear that $\Phi(\mathcal{L})$ correspond to the subspace of matrices whose first k columns are zero. Thus, $\dim\mathcal{L} = n(n-k)$.

Similarly, $\Phi(\mathcal{R})$ corresponds to the subspace of matrices whose last k rows are zero. Hence, $\dim\mathcal{L} = n(n-k)$.

$\Phi(\mathcal{R}\cap\mathcal{L})$ corresponds to matrices whose first k columns and last k rows are zero, so $\dim(\mathcal{L}\cap\mathcal{R}) =$

$(n-k)(n-k)$.
Finally, $\Phi(\mathcal{L}+\mathcal{R})$ corresponds to matrices whose entries (i,j) vanish if $i > n-k$ **and** $j \leq k$. Hence, this is the whole vector space if and only if $k=0$, that is if and only if $T=0$.

Problem 4.
$\ker(j^*)$ is the subspace of functionals $\varphi : \mathbb{C}^3 \longrightarrow \mathbb{C}$ such that $\varphi(w)=0$ for every $w \in W$.
We can write $\varphi = (a\ b\ c)$, so that $\varphi \in \ker(j^*)$ if and only if $\begin{cases} a+c=0 \\ a+b-2c=0 \end{cases}$.
This implies $\ker(j^*) = \text{span}\{(1\ 1\ -1)\}$.

(Call v_3 the third vector of $\mathcal{B}$, that is $v_3 = e_1$.)
By definition, $v_1^* = (v_1^*(e_1)\ v_1^*(e_2)\ v_1^*(e_3))$, $v_2^* = (v_2^*(e_1)\ v_2^*(e_2)\ v_2^*(e_3))$, $v_3^* = (v_3^*(e_1)\ v_3^*(e_2)\ v_3^*(e_3))$. As $e_1 = v_3$, $e_2 = -2v_1 + v_2 + v_3$ and $e_3 = v_1 - v_3$, we obtain $v_1^* = (0\ -2\ 1)$, $v_2^* = (0\ 1\ 0)$ and $v_3^* = (1\ 1\ -1)$.