

These are the solutions to Midterm Exam 2 of **18.700**, Fall 2006.

Problem 1. (25 points)

Consider $L: \mathbb{R}^5 \rightarrow \mathbb{R}^5$ given by $L = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$.

(18 points) Find characteristic and minimal polynomial of L and determine the primary decomposition. (Hint: determinants are useful but not always necessary.)

$$L = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array} \right), \quad A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P_L = P_A \cdot P_B, \quad P_{L, \min} = \text{LCM}(P_{A, \min}, P_{B, \min})$$

$$\left. \begin{array}{l} A^3 - I = 0 \Rightarrow P_{A, \min}(t) \mid t^3 - 1 \\ A(e_1 + e_2 + e_3) = e_1 + e_2 + e_3 \\ A \neq I \end{array} \right\} \Rightarrow P_{A, \min}(t) = t^3 - 1 \Rightarrow P_A(t) = 1 - t^3$$

$$B^2 - I = 0 \text{ and } B \neq \pm I \Rightarrow P_{B, \min}(t) = P_B(t) = t^2 - 1$$

$$B(e_4 + e_5) = e_4 + e_5 \quad B(e_4 - e_5) = -(e_4 - e_5)$$

$$P_L = (t+1)(1-t)^2(1+t+t^2); \quad P_{L, \min} = (t-1)(t+1)(1+t+t^2)$$

because $\mu_{\text{alg}, L}(1) = 2 = \mu_{\text{geom}, L}(1)$.

$$E_1 = \text{span}\{e_4 + e_5, e_1 + e_2 + e_3\}, \quad E_{-1} = \text{span}\{e_4 - e_5\}$$

$$L^2 + L + I = \left(\begin{array}{ccc|cc} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{array} \right) \Rightarrow \ker(L^2 + L + I) = \text{span}\{e_1 - e_2, e_2 - e_3\}$$

$$\mathbb{R}^5 = E_1 \oplus E_{-1} \oplus \ker(L^2 + L + I)$$

3+4 points) Is L triangulizable in $\mathcal{M}_{5 \times 5}(\mathbb{R})$? Is L diagonalizable in $\mathcal{M}_{5 \times 5}(\mathbb{C})$?

L is not triangulizable over \mathbb{R} because $(t^2 + t + 1) \mid P_L$ and $(t^2 + t + 1)$ has no root in \mathbb{R} .

L is diagonalizable over \mathbb{C} . In fact, all roots of P_L are distinct, except 1. But $\mu_{\text{alg}, L}(1) = 2 = \mu_{\text{geom}, L}(1)$, so we have a basis of \mathbb{C}^5 made of eigenvectors of L .

Problem 2. (25 points)

(15 points) Consider the set S of matrices A in $M_{2 \times 2}(\mathbb{R})$ such that $\text{tr}(A) = 0$ and $\text{tr}(A^2) = 8$. Define $A \sim B$ if A is similar to B . How many equivalence classes are there in S ?

Consider A as a complex matrix.

Then A is triangulable with $\{e_1, e_2\}$ on the diagonal. Then $\text{tr}(A) = e_1 + e_2 = 0$.

A^2 will have $\{e_1^2, e_2^2\}$ on the diagonal, so $\text{tr}(A^2) = e_1^2 + e_2^2 = 2e_1^2 = 8 \Rightarrow e_1 = \pm 2, e_2 = -e_1$.

But $\{e_1, e_2\}$ are the roots of P_A , so P_A has distinct real roots ± 2 and so

A is similar to $\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$. Hence there is only 1 class in S .

(10 points) Let X, Y be invertible 2×2 matrices with \mathbb{Q} coefficients such that $X^{-1} = -X$ and $Y^{-1} = -Y$. Show that X is similar to Y .

(Hint: find bases B and C of \mathbb{Q}^2 such that $M_B^B(X) = M_C^C(Y)$.)

$$X^{-1} = -X \Rightarrow (\text{multiplying by } X) \quad X^2 + I = 0.$$

$$Y^{-1} = -Y \Rightarrow Y^2 + I = 0; \text{ so } X, Y \text{ have no eigenvectors in } \mathbb{Q}^2.$$

Pick $0 \neq v \in \mathbb{Q}^2$.

Define $B = \{v, X(v)\}$, $C = \{v, Y(v)\}$.

Clearly, these are bases because X, Y have no ~~invertible~~ eigenvectors and $v \neq 0$.

$$M_B^B(X) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad M_C^C(Y) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

so X, Y are both similar to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

and so they are similar to each other.

Problem 3. (25 points)

Let $A \in M_{n \times n}(\mathbb{F})$ be a fixed matrix and let $L: M_{n \times n}(\mathbb{F}) \rightarrow M_{n \times n}(\mathbb{F})$ be an endomorphism of the vector space $M_{n \times n}(\mathbb{F})$ defined by $L(X) = AX$. Prove the following assertions.

(10 points) L is injective $\iff A$ is invertible.

If A is invertible, then $L(X) = AX = 0 \implies X = A^{-1} \cdot 0 = 0 \implies L$ injective.

If A is not invertible, then $\exists v \in \mathbb{F}^n, v \neq 0$, such that

$A \cdot v = 0$. Take the matrix $X = \begin{pmatrix} (v) & (v) & \dots & (v) \end{pmatrix}$ with

n columns equal to v . Clearly $X \neq 0$

Then $AX = \begin{pmatrix} (Av) & (Av) & \dots & (Av) \end{pmatrix} = 0$, so

that $0 \neq X \in \ker(L)$ and L is not invertible.

(15 points) e is an eigenvalue for $L \iff e$ is an eigenvalue for A . (Hint: construct eigenvectors.)

If e is an eigenvalue for A , $\exists v \in \mathbb{F}^n, v \neq 0$, such that

$$Av = e \cdot v.$$

Take $X = \begin{pmatrix} (v) & \dots & (v) \end{pmatrix} \neq 0$. Then $L(X) = AX =$

$$= \begin{pmatrix} (Av) & \dots & (Av) \end{pmatrix} = \begin{pmatrix} (ev) & \dots & (ev) \end{pmatrix} = e \cdot X \implies X \text{ is eigenvector for } L \text{ with eigenvalue } e.$$

If e is an eigenvalue for L , $\exists 0 \neq X \in M_{n \times n}(\mathbb{F}) : L(X) = eX$.

If $X = \begin{pmatrix} (C_1) & (C_2) & \dots & (C_n) \end{pmatrix}$ where C_k is the k -th column of X , there exists a k such that $C_k \neq 0$ (because $X \neq 0$). Let $v = C_k \neq 0$.

Then $L(X) = AX = \begin{pmatrix} (AC_1) & \dots & (AC_n) \end{pmatrix} = e \begin{pmatrix} (C_1) & \dots & (C_n) \end{pmatrix}$, so that

$A \cdot C_i = e \cdot C_i \forall i$ and in particular $A \cdot v = e \cdot v$.

So e is an eigenvalue for A .

Problem 4. (25 points)

Let V be a vector space over \mathbb{F} of dimension n and let $f : V \rightarrow V$ be a diagonalizable endomorphism. Moreover, let $W \subseteq V$ be an f -invariant subspace and let $f_W : W \rightarrow W$ the restriction of f to W .

(10 points) Prove that the minimal polynomial $p_{f_W, \min}$ of f_W divides the minimal polynomial $p_{f, \min}$ of f .

$$p_{f, \min}(f) = 0 \quad \Rightarrow \quad p_{f, \min}(f) \cdot W = 0 \quad \forall w \in W \Rightarrow$$

$$\Rightarrow p_{f, \min}(f_W) = 0 \quad \Rightarrow \quad p_{f_W, \min} \mid p_{f, \min}.$$

$$\left(\begin{array}{c} \Downarrow \\ p_{f, \min} \in I_{f_W} \\ \Uparrow \end{array} \right)$$

(15 points) Prove that W has a basis of eigenvectors for f and deduce that f_W is diagonalizable.

$$f \text{ diagonalizable} \Rightarrow p_{f, \min}(t) = (t - e_1) \dots (t - e_k)$$

with e_1, \dots, e_k eigenvalues of V .

$\Rightarrow p_{f_W, \min}(t)$ is completely factorized and contains no squares, i.e.

$$p_{f_W, \min}(t) = (t - e_{i_1}) \dots (t - e_{i_m})$$

with $\{i_1, \dots, i_m\} \subseteq \{1, 2, \dots, k\}$.

$\Rightarrow f_W$ is diagonalizable; let $\{w_1, \dots, w_d\} \subseteq W$ be a basis of eigenvectors for f_W , where $d = \dim_{\mathbb{F}} W$.

Then $f_W(w_i) = f(w_i) = \text{multiple of } w_i$, so that w_i is also an eigenvector for f .