These are the solutions to Midterm Exam 1 of **18.700**, Fall 2006.

Problem 1. *(25 points)*

Let $\mathbb{F}$ be a field. For every $\lambda \in \mathbb{F}$, call $M = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{F})$.

Define $\mathcal{S}_\lambda \subseteq \mathcal{M}_{2\times 2}(\mathbb{F})$ to be $\mathcal{S}_\lambda := \{A \in \mathcal{M}_{2\times 2}(\mathbb{F}) \mid AM = MA\}$, i.e. the subset of 2×2 matrices A with coefficients in $\mathbb{F}$ such that $AM = MA$.

For every value $\lambda \in \mathbb{F}$, show that $\mathcal{S}_\lambda$ is a vector subspace of $\mathcal{M}_{2\times 2}(\mathbb{F})$, find a basis of $\mathcal{S}_\lambda$ and compute its dimension.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$AM = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} a\lambda & a+b\lambda \\ c\lambda & c+d\lambda \end{pmatrix}$$

$$MA = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a\lambda+c & b\lambda+d \\ c\lambda & d\lambda \end{pmatrix}$$

$$AM = MA \iff c = 0 \quad \text{and} \quad a = d.$$

So $\quad S_\lambda = \left\{ A \in \mathcal{M}_{2\times 2}(\mathbb{F}) \,\middle|\, A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \right\} \quad \forall \lambda \in \mathbb{F}.$

S_λ is clearly a subspace of $\mathcal{M}_{2\times 2}(\mathbb{F})$, because

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} + \begin{pmatrix} a' & b' \\ 0 & a' \end{pmatrix} = \begin{pmatrix} a+a' & b+b' \\ 0 & a+a' \end{pmatrix}.$$

A basis for S_λ is $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$.

Problem 2. *(25 points)*
Let V, W be vector spaces of finite dimension (over some fixed field $\mathbb{F}$).
Let $M, N : V \longrightarrow W$ be two homomorphisms of vector spaces
and let $(M+N) : V \longrightarrow W$ be their sum.
Prove that $|r-s| \leq \dim \operatorname{Im}(M+N) \leq (r+s)$, where $r = \dim \operatorname{Im}(M)$ and $s = \dim \operatorname{Im}(N)$.
For every $r, s \geq 1$, exhibit an example of (V, W, M, N) for which $|r-s| = \dim \operatorname{Im}(M+N)$
and an example of (V, W, M, N) for which $\dim \operatorname{Im}(M+N) = r+s$.
(Try to prove the upper bound first.)

$\operatorname{Im}(M+N) \subseteq \operatorname{Im}(M) + \operatorname{Im}(N)$, because $(M+N)(v) = M(v) + N(v)$ and $M(v) \in \operatorname{Im}(M)$, $N(v) \in \operatorname{Im}(N)$ $\forall v \in V$.

So $\dim \operatorname{Im}(M+N) \leq \dim \operatorname{Im}(M) + \dim \operatorname{Im}(N) = r+s$.

On the other hand, $\operatorname{Im}(M) \subseteq \operatorname{Im}(M+N) + \operatorname{Im}(-N)$,
so we get $\dim \operatorname{Im}(M) = r \leq \dim \operatorname{Im}(M+N) + s$,
i.e. $\dim \operatorname{Im}(M+N) \geq r-s$.
Similarly, we obtain $\operatorname{Im}(N) \subseteq \operatorname{Im}(M+N) + \operatorname{Im}(-M)$,
$\Rightarrow \dim \operatorname{Im}(M+N) \geq s-r$. $\Rightarrow \dim \operatorname{Im}(M+N) \geq |r-s|$.

Let $V = W = \mathbb{F}^{r+s}$.
[De]fine $M(e_i) = \begin{cases} e_i & \text{if } i \leq r \\ 0 & \text{if } i > r \end{cases}$

$N(e_i) = \begin{cases} 0 & \text{if } i \leq r \\ e_i & \text{if } i > r \end{cases}$

$M+N = \mathrm{Id}_{\mathbb{F}^{r+s}} \Rightarrow$
$\Rightarrow \dim \operatorname{Im}(M+N) = r+S$

Still $V = W = \mathbb{F}^{r+s}$.
Suppose $r \geq s$ (otherwise it is similar).
Define $M(e_i) = \begin{cases} e_i & \text{if } i \leq r \\ 0 & \text{if } i > r \end{cases}$

$N(e_i) = \begin{cases} -e_i & \text{if } i \leq s \\ 0 & \text{if } i > s \end{cases}$

$(M+N)(e_i) = \begin{cases} 0 & \text{if } i \leq s \\ e_i & \text{if } s < i \leq r \\ 0 & \text{if } i > r \end{cases}$

hence $\dim \operatorname{Im}(M+N) = r-s = |r-s|$.

Problem 3. *(25 points)*
Let $x \in \mathbb{Q}$ be a fixed rational number and let $V = \mathbb{Q}[t]_{\leq 3}$ be the vector space (over $\mathbb{Q}$) of polynomials in the variable t of degree at most 3 with rational coefficients.
Define the homomorphism of vector spaces $T : V \longrightarrow V$ as

$$T(p(t)) = p(x) + p'(x)t + p''(x)\frac{t^2}{2}$$

Write the matrix $M_{\mathcal{B}}^{\mathcal{B}}(T)$ associated to T with respect to the (ordered) basis $\mathcal{B} = \{1, t, t^2, t^3\}$.
Determine a basis of $\operatorname{Im}(T^*)$, where $T^* : V^* \longrightarrow V^*$ is the homomorphism dual to T.

$T(1) = 1, \quad T(t) = x + t, \quad T(t^2) = x^2 + 2xt + t^2$

$T(t^3) = x^3 + 3x^2 t + 3xt^2.$

$$M_{\mathcal{B}}^{\mathcal{B}}(T) = \begin{pmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 1 & 3x \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$M_{\mathcal{B}^*}^{\mathcal{B}^*}(T^*) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ x^2 & 2x & 1 & 0 \\ x^3 & 3x^2 & 3X & 0 \end{pmatrix}$$

$\mathcal{B}^* = \{c_0^*, c_1^*, c_2^*, c_3^*\}$ where $c_i^*(a_0 + a_1 t + a_2 t^2 + a_3 t^3) = a_i.$

Thus, a basis for $\operatorname{Im} T^*$ is

$\{ c_0^* + x c_1^* + x^2 c_2^* + x^3 c_3^* = ev_x ,$

$c_1^* + 2x c_2^* + 3x^2 c_3^* = ev_x \circ D ,$

$c_2^* + 3x c_3^* = ev_x \circ D^2 \}$,

where $D(p(t)) = p'(t)$.

Problem 4. *(25 points)*
For every $k \in \mathbb{C}$, find all the homomorphisms of complex vector spaces $L_k : \mathbb{C}^3 \longrightarrow \mathbb{C}^2$ such that $L_k\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}8\\4\end{pmatrix}$, $L_k\begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}-4\\-2\end{pmatrix}$ and $L_k\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}2-k\\2\end{pmatrix}$.
For each of these homomorphisms L_k that you have found, determine bases for $\ker(L_k)$ and $\mathrm{Im}(L_k)$ and write down the matrix associated to L_k.

$\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix}\right\}$ is a basis of $\mathbb{C}^3$, hence there exists a <u>unique</u> L_k for every $k \in \mathbb{C}$.

$$e_1 = \frac{1}{2}\left(\begin{pmatrix}1\\0\\1\end{pmatrix} + \begin{pmatrix}1\\0\\-1\end{pmatrix}\right)$$

$$e_2 = -\begin{pmatrix}1\\-1\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

$$e_3 = \frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\0\\-1\end{pmatrix}.$$

$$L(e_1) = \frac{1}{2}L\begin{pmatrix}1\\0\\1\end{pmatrix} + \frac{1}{2}L\begin{pmatrix}1\\0\\-1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}8\\4\end{pmatrix} + \frac{1}{2}\begin{pmatrix}-4\\-2\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}.$$

$$L(e_2) = -L\begin{pmatrix}1\\-1\\0\end{pmatrix} + \frac{1}{2}L\begin{pmatrix}1\\0\\1\end{pmatrix} + \frac{1}{2}L\begin{pmatrix}1\\0\\-1\end{pmatrix} = -\begin{pmatrix}2-k\\2\end{pmatrix} + \begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}k\\-1\end{pmatrix}.$$

$$L(e_3) = \frac{1}{2}L\begin{pmatrix}1\\0\\1\end{pmatrix} - \frac{1}{2}L\begin{pmatrix}1\\0\\-1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}8\\4\end{pmatrix} - \frac{1}{2}\begin{pmatrix}-4\\-2\end{pmatrix} = \begin{pmatrix}6\\3\end{pmatrix}.$$

$$L = \begin{pmatrix}2 & k & 6\\1 & -1 & 3\end{pmatrix}.$$

$L(e_3) = 3L(e_1) \Rightarrow e_3 - 3e_1 \in \ker L_k$.

$\begin{pmatrix}k\\-1\end{pmatrix}$ is proportional to $\begin{pmatrix}2\\1\end{pmatrix} \Leftrightarrow k = -2$.

If $k \neq -2$, then $\ker L_k$ has basis $\left\{\begin{pmatrix}-3\\0\\1\end{pmatrix}\right\}$ and $\mathrm{Im}\, L_k$ has basis $\left\{\begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}k\\-1\end{pmatrix}\right\}$.

If $k = -2$, $\ker L_k$ has basis $\left\{\begin{pmatrix}-3\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$ and $\mathrm{Im}\, L_k$ has basis $\left\{\begin{pmatrix}2\\1\end{pmatrix}\right\}$.