These are the solutions to Final Exam of **18.700**, held on December 21st, 2006.

Problem 1. *(20 points)*

(15 points) Let $\mathcal{X}$ be the set of 3×3 matrices with real coefficients A such that $A^5 = A^4$. Classify the matrices in $\mathcal{X}$ up to similitude.

$$A^5 = A^4 \Rightarrow q(A) = 0 \text{ for } q = t^4(t-1).$$

$$\Rightarrow p_{min,A}(t) \mid q(t).$$

(1) $p_{min,A} = t \Rightarrow A = 0.$

(2) $p_{min,A} = t^2 \Rightarrow A \overset{sim.}{\sim} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(3) $p_{min,A} = t^3 \Rightarrow A \overset{sim}{\sim} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

(4) $p_{min,A} = t-1 \Rightarrow A = I.$

(5) $p_{min,A} = t(t-1) \Rightarrow \begin{cases} A \overset{sim}{\sim} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ ; or} \\ A \overset{sim.}{\sim} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{cases}$

(6) $p_{min,A} = t^2(t-1) \Rightarrow A \overset{sim.}{\sim} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(5 points) Find all the matrices $B \in \mathcal{X}$ that also satisfy $B^2 + B = 2I$.

$B^2 + B - 2I = 0 \Rightarrow r(B) = 0$

for $r(t) = t^2 + t - 2 =$

$= (t+2)(t-1)$.

$p_{min,B} \mid r(t) = (t+2)(t-1)$

$p_{min,B} \mid q(t) = t^4(t-1)$

$\Rightarrow p_{min,B} = t-1 \Rightarrow \boxed{B = I.}$

(12 points) Let $A \in O(3,\mathbb{R}) = \{N \in \mathcal{M}_{3\times 3}(\mathbb{R}) \mid {}^tNN = I\}$ be a 3×3 real orthogonal matrix such that $\det(A) = 1$ and $\operatorname{tr}(A) = -1$. Show that A is symmetric.

By classification of orthogonal matrices, if $\det A = 1$, then $\exists\, N \in O(3,\mathbb{R})$:

$${}^tNAN = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \Rightarrow \operatorname{tr}(A) = 1 + 2\cos\theta.$$

$\| \quad N^{-1}AN$

$\operatorname{tr}(A) = -1 \Rightarrow \cos\theta = -1 \Rightarrow \theta = \pi.$

So $${}^tNAN = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow A \text{ symmetric}$$ (because congruent to ${}^tNAN$ which is symmetric).

(8 points) Show that a 3×3 real orthogonal matrix $A \in O(3,\mathbb{R})$ is the identity if and only if $A - I$ is nilpotent (i.e. $(A-I)^k = 0$ for some positive integer k).

If $A = I$, then $A - I = 0$ and so $(A-I)$ is nilpotent.

Vice versa, let $A - I$ be nilpotent, so that $+1$ is the only eigenvalue of A.

By classification of orthogonal matrices, $\exists\, N \in O(3,\mathbb{R})$: ${}^tNAN = N^{-1}AN =$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \text{ so that}$$

$$N^{-1}(A-I)N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos\theta - 1 & -\sin\theta \\ 0 & \sin\theta & \cos\theta - 1 \end{pmatrix}.$$

$A - I$ nilpotent $\Rightarrow N^{-1}(A-I)N$ nilpotent $\Rightarrow \operatorname{tr}(N^{-1}(A-I)N) = 0$

$\Rightarrow \cos\theta = 1 \Rightarrow \theta = 0 \Rightarrow A = I.$

Problem 3. *(20 points)*
Let $V = \mathbb{R}[t]_{\leq 3}$ be the vector space of polynomials with degree at most 3 and real coefficients and let $T : V \longrightarrow V$ be the endomorphism defined as $T(p(t)) = p(2t+1) - 2p(t)$.

(8 points) Write the matrix $M_{\mathcal{C}}^{\mathcal{C}}(T)$ that represents T with respect to the basis $\mathcal{C} = \{1, t, t^2, t^3\}$ of V.

$$T(1) = 1 - 2 = -1$$
$$T(t) = 2t + 1 - 2t = 1$$
$$T(t^2) = (2t+1)^2 - 2t^2 = 4t^2 + 4t + 1 - 2t^2 = 2t^2 + 4t + 1$$
$$T(t^3) = (2t+1)^3 - 2t^3 = 8t^3 + 12t^2 + 6t + 1 - 2t^3 = 6t^3 + 12t^2 + 6t + 1$$

$$M_{\mathcal{C}}^{\mathcal{C}}(T) = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & 2 & 12 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$

(4 points) Compute characteristic and minimal polynomial of T.

$$p_T(t) = (t+1)\,t\,(t-2)(t-6).$$

$$p_{T,\min}(t) = (t+1)\,t\,(t-2)(t-6).$$

points) Determine an explicit basis $\mathcal{B}$ of V such that $M_{\mathcal{B}}^{\mathcal{B}}(T)$ is in Jordan form. Write down $M_{\mathcal{B}}^{\mathcal{B}}(T)$.

$\underline{\ker(T)} = \text{span}\{t+1\}$; define $v_0 = t+1$.

$\underline{\ker(T+I)} = \text{span}\{1\}$; define $v_{-1} = 1$.

$\underline{\ker(T-2I)}$: $M_e^e(T-2I) = \begin{pmatrix} -3 & 1 & 1 & 1 \\ 0 & -2 & 4 & 6 \\ 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 4 \end{pmatrix}$

$\quad \shortparallel$

$\text{span}\{t^2+2t+1\}$

Define $v_2 = t^2+2t+1$.

$\underline{\ker(T-6I)}$: $M_e^e(T-6I) = \begin{pmatrix} -7 & 1 & 1 & 1 \\ 0 & -6 & 4 & 6 \\ 0 & 0 & -4 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\quad \shortparallel$

$\text{span}\{t^3+3t^2+3t+1\}$.

Define $v_6 = t^3+3t^2+3t+1$.

Jordan basis $\mathcal{B} = \{v_{-1}, v_0, v_2, v_6\}$:

$$M_{\mathcal{B}}^{\mathcal{B}}(T) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}.$$

Problem 4. *(20 points)*
Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ be a skew-symmetric $n \times n$ real matrix (i.e. ${}^tA = -A$).

(10 points) Prove that A^2 is symmetric and semi-negative-definite (i.e. the scalar product b defined by A^2 on $\mathbb{R}^n$ satisfies $b(v,v) \leq 0\ \forall v \in \mathbb{R}^n$).

Let $v \in \mathbb{R}^n$. Then ${}^t(A^2) = {}^tA \cdot {}^tA = (-A)^2 = A^2$, so that A^2 is symmetric and

$${}^tv A^2 v = -{}^tv\, {}^tA A v = -{}^t(Av)(Av) \leq 0.$$

(10 points) Prove that A is diagonalizable over $\mathbb{R}$ if and only if $A = 0$.

$A=0$ is already diagonal.

Conversely, suppose A diagonalizable, so that

$$\exists N \in GL(n,\mathbb{R}) : \quad NAN^{-1} = D = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}, a_i \in \mathbb{R}.$$

Thus, $N(A^2)N^{-1} = D^2 = \begin{pmatrix} a_1^2 & & 0 \\ & \ddots & \\ 0 & & a_n^2 \end{pmatrix}$, $a_i^2 \geq 0$.

By the previous part, A^2 is symmetric and semi-negative, so that all its eigenvalues are ≤ 0. This implies that $a_1 = a_2 = \dots = a_n = 0$ and so $D = A = 0$. ■

Problem 5. *(20 points)*
Let V be a vector space over $\mathbb{F}$ of dimension $n > 1$, let $0 \neq \varphi \in V^*$ be a nonzero homomorphism $\varphi : V \longrightarrow \mathbb{F}$ and let $0 \neq w \in V$. Consider the homomorphism $L : V \longrightarrow V$ defined as $L(v) = v + \varphi(v)w$.

(10 points) Assume $w \notin \ker(\varphi)$. What is the Jordan form of a matrix representing L?

Take $\{u_1 \dots u_{n-1}\}$ basis of $\ker\varphi$, so that $\mathcal{B} = \{w\ u_1 \dots u_{n-1}\}$ is a basis of V.

$$M^{\mathcal{B}}_{\mathcal{B}}(L) = \left(\begin{array}{c|c} 1+\varphi(w) & 0 \cdots 0 \\ \hline 0 & \underbrace{\begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix}}_{n-1} \\ \vdots & \\ 0 & \end{array}\right) = \left(\begin{array}{c|c} 1+\varphi(w) & 0 \\ \hline 0 & I_{n-1} \end{array}\right)$$

and this is the Jordan form.

(10 points) Assume $w \in \ker(\varphi)$. What is the Jordan form of a matrix representing L?

Take $\{u_1 \dots u_{n-2}\, w\}$ basis of $\ker \varphi$ and $v \notin \ker \varphi$. So that $\mathcal{B} = \{w\, u_1 \dots u_{n-2}\, v\}$ is a basis of V.

$$M_{\mathcal{B}}^{\mathcal{B}}(L) = \left(\begin{array}{c|c} I_{n-1} & \begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \cdots 0 & 1 \end{array}\right)$$

m $\ker\left(M_{\mathcal{B}}^{\mathcal{B}}(L-I)\right) = n-1 \implies p_{L,\min}(t) = (t-1)^2$

and so the Jordan form is

$$J = \left(\begin{array}{c|c} \begin{matrix} 1 & 0 \\ 1 & 1 \end{matrix} & 0 \\ \hline 0 & I_{n-2} \end{array}\right)$$

Problem 6. *(20 points)*
For every $c \in \mathbb{R}$, let $T_c : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the endomorphism represented by the matrix

$$T_c = \begin{pmatrix} 1 & 2 & c \\ 1 & 0 & 1 \\ 2 & 3 & -1 \end{pmatrix}.$$

(5 points) Compute the determinant and the rank of T_c for every value of $c \in \mathbb{R}$.

$$\det(T_c) = -2\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} - 3\begin{vmatrix} 1 & c \\ 1 & 1 \end{vmatrix} = -2(-1-2) - 3(1-c) =$$

$$= 6 - 3 + 3c = 3 + 3c.$$

For $c \neq -1$, $\operatorname{rk}(T_c) = 3$.

For $c = -1$, $T_{-1} = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 2 & 3 & -1 \end{pmatrix}$ and so $\operatorname{rk}(T_{-1}) = 2$ because the first two columns are not proportional.

(8 points) If T_c is not invertible, find a basis for its kernel and its image.

T_c not invertible $\Longleftrightarrow$ $c = -1$.

A basis for the image is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \right\}$.

For the kernel, we row-reduce $T_{-1} = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 2 & 3 & -1 \end{pmatrix} \rightsquigarrow$

$$\rightsquigarrow \begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 2 \\ 0 & -1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

row-reduced form.

$$0 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+z \\ y-z \\ 0 \end{pmatrix} \Longleftrightarrow \begin{array}{l} x = -z \\ y = z. \end{array}$$

Basis for $\ker(T_{-1})$ is $\left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \right\}$.

(7 points) For $c = 0$ and for $c = -1$, find the set of solutions of the linear system $T_c X = Y$, where $X \in \mathbb{R}^3$ and $Y = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} \in \mathbb{R}^3$.

For any c, $2\begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + (-1)\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$, so that $T_c \begin{pmatrix} -1 \\ +2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$.

For $c=0$, T_0 is invertible and so $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ is the unique solution.

For $c=-1$, $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ is a particular solution.

The general solution is $\begin{pmatrix} -1 \\ +2 \\ 0 \end{pmatrix} + \alpha \cdot \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ with $\alpha \in \mathbb{R}$,

so that solutions $= \left\{ \begin{pmatrix} \alpha - 1 \\ 2 - \alpha \\ -\alpha \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, \alpha \in \mathbb{R} \right\}$.