These are the solutions to Final Exam of **18.700**, held on December 19th, 2006.

(5 points) Show that two 2×2 real matrices A, B such ... each other.

Let $0 \neq v \in \mathbb{R}^2$.

As $A^2 + I = 0$, the minimal polynomial is $p_{A,min} = t^2 + 1$ and so A has no eigenvectors.

Hence $w := Av$ is not proportional to v and $\mathcal{B} = \{v, w\}$ is a basis.

$M_{\mathcal{B}}^{\mathcal{B}}(A) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, because $Aw = A(Av) = A^2 v = -v$.

Similarly, define $z = Bv$ and $\mathcal{C} = \{v, z\}$.

Then $M_{\mathcal{C}}^{\mathcal{C}}(B) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and so both A and B are similar to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and so are similar to each other.

(15 points) Let $\mathcal{X}$ be the set of 3×3 matrices with real coefficients A such that $A^4 = I$. Classify the matrices in $\mathcal{X}$ up to similitude.

$A^4 = I \Rightarrow q(A) = 0$ with $q(t) = t^4 - 1$.

$q(t) = t^4 - 1 = (t^2 - 1)(t^2 + 1) = (t-1)(t+1)(t^2+1)$.

$p_{min,A}(t) \mid q(t)$.

(1) $p_{min,A} = t - 1 \Rightarrow A = I$

(2) $p_{min,A} = t + 1 \Rightarrow A = -I$.

(3) $p_{min,A} = (t-1)(t+1)$

$\Rightarrow A \overset{sim.}{\sim} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ or

$A \overset{sim.}{\sim} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

$$\Rightarrow A \overset{\text{sim.}}{\sim} \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & -1 \\ 0 & 1 & 0 \end{array}\right)$$

(5) $p_{\min,A} = (t^2+1)(t+1)$

$$\Rightarrow A \overset{\text{sim.}}{\sim} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

<u>Remark</u>. $p_{\min,A} = (t^2+1)$ is ruled out because p_A cannot be either (t^2+1) or $(t^2+1)^2$.

/ *(14 points)* Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ be an $n \times n$ matrix with real entries such that $\mathrm{rk}(A) = k \leq n$. Show that there exist matrices $P \in \mathrm{GL}(n,\mathbb{R})$ invertible and $Q \in \mathrm{O}(n,\mathbb{R})$ orthogonal such that $PAQ = \left(\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0_{n-k} \end{array}\right)$.

Let $V = W \oplus \ker(A)$, where $\dim W = k$ and W is any <u>orthogonal</u> complement of $\ker(A)$.

Let $\{u_1 \dots u_{n-k}\}$ be an orthonormal basis of $\ker A$ and $\{w_1 \dots w_k\}$ be an orthonormal basis of W.

Then $\mathcal{B} = \{w_1 \dots w_k\ u_1 \dots u_{n-k}\}$ is an orthonormal basis of V.

Define Q = endomorphism that sends $\{e_1 \dots e_n\}$ to $\mathcal{B}$.

Let $V = Z \oplus \mathrm{Im}(A)$, where Z is any complement.

Let $\{z_1 \dots z_{n-k}\}$ be a basis of Z so that $\mathcal{C} = \{Aw_1, \dots, Aw_k, z_1, \dots, z_{n-k}\}$ is a basis of V.

Let P = endomorphism that sends $\mathcal{C}$ to $\{e_1 \dots e_n\}$.

(6 points) Using the result above (whether or not you proved it), show that $\mathrm{rk}(A\,{}^tA) = \mathrm{rk}(A)$.

Let $PAQ = D = \left(\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0_{n-k} \end{array}\right)$

with $P \in GL(n,\mathbb{R})$ and $Q \in O(n,\mathbb{R})$. 3

Then $(PAQ)({}^t PAQ) = PAQ\,{}^tQ\,{}^tA\,{}^tP =$

$D = D\,{}^tD \qquad = PA\,{}^tA\,{}^tP$

and $\mathrm{rk}(A\,{}^tA) = \mathrm{rk}(PA\,{}^tA\,{}^tP) = \mathrm{rk}(D) = \mathrm{rk}(A)$ 3

Problem 3. *(20 points)*
Let $V = \mathbb{C}[t]_{\leq 3}$ be the vector space of polynomials with degree at most 3 and complex coefficients and let $T : V \longrightarrow V$ be the endomorphism defined as $T(p(t)) = p(t+1) + p(t-1)$.

(8 points) Write the matrix $M^{\mathcal{C}}_{\mathcal{C}}(T)$ that represents T with respect to the basis $\mathcal{C} = \{1, t, t^2, t^3\}$ of V.

$T(1) = 2$
$T(t) = 2t$
$T(t^2) = 2t^2 + 2$
$T(t^3) = 2t^3 + 6t$

$$M^{\mathcal{C}}_{\mathcal{C}}(T) = \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 6 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

(8 points) Compute characteristic and minimal polynomial of T.

$p_T(t) = (t-2)^4.$

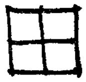
$\dim \ker (T - 2I) = 2.$

$$M^{\mathcal{C}}_{\mathcal{C}}((T-2I)^2) = (M^{\mathcal{C}}_{\mathcal{C}}(T) - 2I)^2 = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^2 = 0.$$

So that $\dim \ker (T-2I)^2 = 4.$

$\Rightarrow \quad p_{T,\min}(t) = (t-2)^2.$

(4 points) Determine an explicit basis $\mathcal{B}$ of V such that $M^{\mathcal{B}}_{\mathcal{B}}(T)$ is in Jordan form. Write down $M^{\mathcal{B}}_{\mathcal{B}}(T)$.



$V = \ker(T-2I) \oplus W$, W = any complement.

We want $\{w_1, w_2\}$ basis of W.

As $\ker(T-2I) = \text{span}\{1, t\}$, we can choose $W = \text{span}\{t^2, t^3\}$ and $w_1 = t^2$, $w_2 = t^3$.

The Jordan basis will be $\mathcal{B} = \{w_1, (T-2I)w_1, w_2, (T-2I)w_2\} =$

$= \{t^2, 2, t^3, 6t\}$.

$$M^{\mathcal{B}}_{\mathcal{B}} = \left(\begin{array}{cc|cc} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right).$$

Let $\mathbb{R}^3$ be endowed with the standard scalar product and let $A : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be an orthogonal transformation.

(15 points) Prove that A is diagonalizable if and only if there exist two A-invariant subspaces $W_1, W_2 \subset \mathbb{R}^3$ such that $\dim W_1 = \dim W_2 = 2$ and $W_1 \neq W_2$.

Clearly, if A is diagonalizable, then $\{v_1, v_2, v_3\}$ being the basis of eigenvectors, we can take $W_1 = \text{span}\{w_1, w_2\}$ and $W_2 = \text{span}\{w_2, w_3\}$.

Conversely, suppose $\exists\ W_1 \neq W_2$ A-inv. of dim. 2.

Then $W_1 \cap W_2 = Z$ is A-invariant and one-dimensional.

$Z = \text{span}\{z\}$. Also $Z^\perp$ is A-invariant and 2-dimensional.

$Z^\perp \cap W_1 = \text{span}\{w_1\}$ is A-invariant and 1-dim.
$Z^\perp \cap W_2 = \text{span}\{w_2\}$ is A-invariant and 1-dimensional.
$\mathcal{B} = \{z, w_1, w_2\}$ is a basis of eigenvectors for A, so A is diagonalizable.

(5 points) Does the thesis still hold true if $A \in \mathcal{M}_{3\times 3}(\mathbb{R})$ is not required to be orthogonal?

No. In fact, $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

is not diagonalizable ($\dim \ker(A - 2I) = 2$!)

and $W_1 = \text{span}\{e_1, e_2\}$ and $W_2 = \text{span}\{e_2, e_3\}$ are A-invariant and $W_1 \neq W_2$.

Problem 5. *(20 points)*
Let V be a real vector space of dimension n and let $f : V \longrightarrow V$ be an endomorphism such that $f^2 = Id$.

(15 points) Does there exist a positive-definite scalar product b on V such that f is self-adjoint with respect to b?

Yes; $f^2 = Id \Rightarrow P_{f,min} \mid (t-1)(t+1)$,

so that $\exists\ \mathcal{B} = \{v_1 \dots v_n\}$ basis of V such that

$$M^{\mathcal{B}}_{\mathcal{B}}(f) = \left(\begin{array}{c|c} I_k & 0 \\ \hline 0 & -I_{n-k} \end{array}\right) \text{ for some } 0 \le k \le n.$$

Define $b(v_i, v_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$.

This defines a symmetric bilinear form b with orthonormal basis $\mathcal{B}$, so b is positive-definite.
To check that f is self-adjoint it is sufficient to notice that $b(f(v_i), v_j) = 0 = b(v_i, f(v_j))$ if $i \neq j$

and $b(f(v_i), v_i) = b(v_i, f(v_i))$ $\forall i$.

(5 points) Is f self-adjoint with respect to every positive-definite scalar product on V?

If $f = I$ or $f = -I$, then it is self-adjoint for every b.
Otherwise, let $f(v) = v$ and $f(w) = -w$.
Take a basis $\{v, v+w, z_1 \dots z_{n-2}\} = \mathcal{C}$ and define a $b(\cdot,\cdot)$ so that $\mathcal{C}$ is orthonormal.
Then $b(v,v) = 1 = b(v+w, v+w)$, $b(v, v+w) = 0$ and $b(v,w) = -1$.

then $0 = b(v, v+w) = b(f(v), v+w)$ but
$b(v, f(v+w)) = b(v, v-w) = b(v,v) - b(v,w) = 2 \neq 0$.
So f is not self-adjoint w.r.t. this b.

Problem 6. *(20 points)*
For every $c \in \mathbb{R}$, let $T_c : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the endomorphism represented by the matrix

$$T_c = \begin{pmatrix} 2 & 0 & -2 \\ 1 & 1 & 0 \\ c & -1 & 1 \end{pmatrix}.$$

(5 points) Compute the determinant and the rank of T_c for every value of $c \in \mathbb{R}$.

$$\det(T_c) = 2\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ c & -1 \end{vmatrix} = 2 - 2(-1-c) = 2 + 2c = 4 + 2c.$$

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For $c \neq -2$, $\operatorname{rk}(T_c) = 3$.

For $c = -2$, $T_{-2} = \begin{pmatrix} 2 & 0 & -2 \\ 1 & 1 & 0 \\ -2 & -1 & 1 \end{pmatrix}$ has rank 2

because the first two columns are not proportional to each other. 2

(8 points) If T_c is not invertible, find a basis for its kernel and its image.

T_c not invertible $\iff$ $\underline{c = -2}$. 2

A Basis of $\operatorname{Im}(T_{-2})$ is $\left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$. 2

For $\ker(T_{-2})$, let's row-reduce T_{-2}

$$T_{-2} = \begin{pmatrix} 2 & 0 & -2 \\ 1 & 1 & 0 \\ -2 & -1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & -2 \\ -2 & -1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & -2 \\ 0 & 1 & 1 \end{pmatrix} \rightsquigarrow$$

$$\rightsquigarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

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$$0 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-z \\ y+z \\ 0 \end{pmatrix} \Leftrightarrow \begin{matrix} x = z \\ y = -z \end{matrix}$$

Basis of $\ker(T_{-2})$ is $\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$.

(7 points) For $c = 0$ and for $c = -2$, find the set of solutions of the linear system $T_c X = Y$, where $X \in \mathbb{R}^3$ and $Y = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \in \mathbb{R}^3$.

$$-1 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} - 1 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \text{ so that}$$

$X = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$ is a particular solution.

If $c = 0$, then T_0 is invertible and so $X = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$ is the unique solution.

If $c = -2$, then the general solution is

$$\begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \text{ so that solutions} = \left\{ \begin{pmatrix} \alpha \\ -1-\alpha \\ \alpha-1 \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, \alpha \in \mathbb{R} \right\}.$$