Analytic algorithms for the moment polytope

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Based on joint work with

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Outline

1. Moment polytopes by example
2. Algorithms for the general problem
Moment polytopes
Motivating question

Horn’s problem:

Are $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}^n$ the spectra of three $n \times n$ matrices $H_1, H_2, H_3$ such that

$$H_1 + H_2 = H_3?$$

If so, can one find the matrices efficiently?
Let $\mathcal{V} = \mathbb{P}(\text{Mat}(n)^2)$, define

$$
\mu : \mathcal{V} \to \text{Herm}(n)^3
$$

by

$$
\mu : [A_1, A_2] \mapsto \left( \frac{A_1 A_1^\dagger}{\|A_1\|^2}, \frac{A_2 A_2^\dagger}{\|A_2\|^2}, \frac{A_1^\dagger A_1 + A_2^\dagger A_2}{\|A_1\|^2 + \|A_2\|^2} \right).
$$

Note $\text{eigs}(AA^\dagger) = \text{eigs}(A^\dagger A)$, so

$$
\text{eigs}(A_1 A_1^\dagger), \text{ eigs}(A_2 A_2^\dagger), \text{ eigs}(A_1^\dagger A_1 + A_2^\dagger A_2)
$$

is a “yes” instance to Horn’s problem (in fact, all such instances take this form).
Moment polytopes

- $G = \text{GL}(n)$
- $\pi : G \rightarrow \mathbb{C}^m$ a representation of $G$ where $U(n)$ acts unitarily
- $\mathcal{V} \subset \mathbb{P}(\mathbb{C}^m)$ a projective variety fixed by $G$,

**Moment map** is the map $\mu : \mathcal{V} \rightarrow n \times n \text{ Hermitians} =: \text{Herm}(n)$ given by

$$\mu : v \mapsto \nabla_{H \in \text{Herm}(n)} \log \| e^H \cdot v \|$$

$i \mu$ is a moment map for $U(n)$ in the physical sense! In particular:

**Theorem (Kirwan)**

Image of

$$\mathcal{V} \xrightarrow{\mu} \text{Herm}(n) \xrightarrow{\text{take eigs.}} \mathbb{R}^n$$

is a convex polytope in $\mathbb{R}^n$ known as **moment polytope**, denoted $\Delta(\mathcal{V})$.
Horn polytope

- \( \mathcal{V} = \mathbb{P}(\text{Mat}(n)^2) \)
- \( G = \text{GL}(n)^3 \)
- \( \pi \) given by
  \[
  (g_1, g_2, g_3) \cdot (A_1, A_2) = (g_1 A_1 g_3^\dagger, g_2 A_2 g_3^\dagger).
  \]
- \( \mu : \mathcal{V} \rightarrow \text{Herm}(n)^3 \) given by
  \[
  \mu : [A_1, A_2] \mapsto \frac{(A_1 A_1^\dagger, A_2 A_2^\dagger, A_1^\dagger A_1 + A_2^\dagger A_2)}{\| A_1 \|^2 + \| A_2 \|^2}.
  \]

Thus, image of

\[
\mathcal{V} \xrightarrow{\mu} \text{Herm}(n)^3 \xrightarrow{\text{take eigs.}} (\mathbb{R}^n)^3
\]

is the* solution set of the Horn problem!
Link to algebra

[CF: Missing!]
Algorithmic tasks

**Input** \((\mathcal{V}, \pi, \lambda)\)

- Projective variety \(\mathcal{V}\) as arithmetic circuit parametrizing it
- Representation \(\pi\) as its list of irreducible subrepresentations as elements of \(\mathbb{Z}^n\)
- Target \(\lambda \in \mathbb{Q}^n\)

1. **membership**: determine whether \(\lambda\) in \(\Delta(\mathcal{V})\).
2. **\(\varepsilon\)-search**: given \(\lambda \in \mathbb{R}^n\), either find an element \(v \in \lambda\) such that
   - \(\|\mu(v) - \text{diag}(\lambda)\| < \varepsilon\), OR
   - correctly declare \(\lambda \notin \Delta(\mathcal{V})\).
   i.e. find an approximate preimage under \(\mu\)!

\(1/\exp(\text{poly})\)-search suffices for membership!
Algorithm for $\varepsilon$-search for Horn polytope (F18)

**Input:** $(\lambda_1, \lambda_2, \lambda_3) \in (\mathbb{R}^n)^3$ and $\varepsilon > 0$.

1. Choose $A_1, A_2$ at random. Define

$$\mu_1 = A_1 A_1^\dagger, \quad \mu_2 = A_2 A_2^\dagger, \quad \mu_3 = A_1^\dagger A_1 + A_2^\dagger A_2.$$

Want $\mu_i = \text{diag}(\lambda_i)$

2. **while** $\|\mu_3 - \text{diag}(\lambda_3)\| > \varepsilon$, **do:**

   a. Choose $B$ upper triangular such that $B^\dagger \mu_3 B = \text{diag}(\lambda_3)$,
   
   **Set** $A_i \leftarrow A_i B$.

   b. For $i \in 1, 2$, choose $B_i$ upper triangular s.t. $B_i^\dagger \mu_i B_i = \text{diag}(\lambda_i)$,
   
   **Set** $A_i \leftarrow B_i^\dagger A_i$.

3. **output** $A_1^\dagger A_1, A_2^\dagger A_2$. 
Complexity of moment polytope membership?

The case $\lambda = 0$ is the null-cone problem from Ankit’s talk!

1. Is membership in $P$?
   - For tori ($G = \mathbb{C}^n$) Folklore, [SV17]
   - For Horn polytope, by saturation conjecture [MNS12]

2. Is it in $RP$?
   - We think so in general, but no proof yet!

3. Is it in $NP$ or $coNP$?
   - In $NP \cap coNP$ for $\mathcal{V} = \mathbb{P}(\mathbb{C}^m)$ [BCMW17]
   - Not known in general!
General algorithms
Convert $\epsilon$-search to an optimization problem

[CF: MISSING!]
Optimization algorithms

**Alternating minimization:** $\text{poly}(1/\varepsilon)$ time [BFGOWW18]

- Tensor products of easy reps e.g. Horn, $k$-tensors

$\log \text{cap}_\lambda(v)$ can be cast as a **geodesically convex program**!

Domain is positive-semidefinite matrices; geodesics through $P$ take the form $\sqrt{P} e^{Ht} \sqrt{P}$

**Geodesic gradient descent:** $\text{poly}(1/\varepsilon)$ time [BFGOWW19]

- Any representation, e.g. $\mathcal{V} = \bigwedge^k \mathbb{C}^n, \text{Sym}^k \mathbb{C}^n$, arbitrary quivers

**Geodesic trust-regions:** $\text{poly}(\log(1/\varepsilon), \log \kappa)$ time [BFGOWW19]

- $\kappa$ is smallest condition-number of an $\varepsilon$-optimizer for $\text{cap}_\lambda(v)$
- Polynomial for some interesting cases, e.g. arbitrary quivers with $\lambda = 0$
1. Is moment polytope membership in $\text{NP} \cap \text{coNP}$, or even $\text{RP}$ or $\text{P}$?
2. Membership is in $\text{P}$ for Horn’s problem. But how about $\exp(-\text{poly})$-search?
3. If $(A_1, A_2)$ a random pair of matrices, does $\text{cap}_\lambda(A_1, A_2)$ have an $\varepsilon$-minimizer with condition number at most $\exp(\text{poly}(\log(1/\varepsilon), \langle \lambda \rangle))$?
Merci!