Lecture 9

Plan: 1) Finish polyhedra
2) Preview applications.

Polyhedra Cont.

Recall: Nonredundant = Facets.

- Inequality \( a_i^T x \leq b_i \) is redundant if \( P \) unchanged when it's removed.

- \( I^- := \{ i : \forall x \in P, a_i^T x = b_i \} \) "equalities"

- \( \subset \) \( P \) \( a_i^T x < b_i \)
I_\prec := \{ x \in \mathbb{R}^n : y_i \prec x \mid i \in I \prec \}

"real inequalities"

e.g.

\[ P = \left\{ x : \begin{array}{l}
x_1 + x_2 \leq 1 \\
-x_1 \leq 0 \\
-x_2 \leq 0 \\
x_3 \leq 0 \\
-x_3 \leq 0
\end{array} \right\} \]

THEN:

Not facet \implies \text{redundant.}

\[ \text{face } a_i^T x = b_i \text{ for } i \in I_\prec \text{ not facet} \]

\[ \implies a_i^T x \leq b_i \text{ is redundant.} \]

(need \( i \in I_\prec \), e.g. \( x_3 \geq 0, x_3 \leq 0 \) in example neither facets nor redundant)
Facet $\Rightarrow$ non-redundant.

$F$ is a facet of $P$, $\Rightarrow$

$\exists i \in I_\prec$ s.t. $F$ from $a_i^T x = b_i$.

**Take-home:** in minimal description of $P$, need

- lin-indep set of equalities ($I_E$)
- one inequality per facet ($I_\prec$).

**Proof** We only prove $\Rightarrow$.

- Suppose $a_i^T x \leq b_i$ not redundant
  
  $\Rightarrow$
want to show correspondence face to face.

- We'll do this by showing
  \[ \dim(F) \geq \dim(P) - 1 \]
  \& \ [\dim(F) \neq \dim(P)].

\[ \dim(F) \geq \dim(P) - 1 \]

- \( x \Rightarrow \text{Is } \land \text{s.t.} \)

\[ a_i^x > b_i \]

but \[ a_i^x \leq b_j \ \forall j \neq i. \]

e.g. \( i = 4 \)
• Let $F_i$ be face $a_i^T x = b_i$.
• $\forall x \in P$, line segment $x \rightarrow x_0$ has unique $x_0 \in F_i$.

$$\text{e.g. } i = 4$$

$\Rightarrow$ any point $x \in P$ contained in $\text{aff}(F_i, x_0)$!

• $P \subseteq \text{aff}(F_i, x_0) \Rightarrow \dim(P) \leq \dim(F_i) + 1$. 
Recall: Near vertex $= \text{Cone(polytope)}$

(N.Y.C. Theorem)
Let $v_0$ vertex of $P$ from valid inequality $c^T x \leq m$. Let $E$ be such that $c^T v' \leq m - 3$ for all other vertices $v'$.

Then

$$P_0 = \{ x \in P : c^T x = m - 3 \}$$

is a polytope & is bijection

$$\{ P_0 \text{'s dim $k$ faces} \} \leftrightarrow \{ -1, 1, \ldots, k+1 \ \text{faces} \}$$
Corollary: Graph connected

Graph of vertices & edges of polyhedron $P$ is always connected.

In particular: if $v^* \text{ max. } c^T x$ over $P$,

$\exists v_0 \rightarrow v^*$ path which doesn't decrease objective.
Proof of Corollary:

- Suppose \( u^* \) unique max of \( C^T X \) over \( P \).

- Enough to show that

  \( \forall \) vertices \( V_0 \neq P \),

  \( \exists \) edge to vertex \( v_1, w \) with

  \[ C^T v_1 > C^T v_0. \]

  (by finiteness of # vertices).

![Diagram showing vertices and edges]
• Let $P_0$ be polytope from last theorem.

• Let $\mathbf{x}$ be intersection of $P_0$ and segment joining $V_0, V^*$. 
• note that \( c^T v_0 < c^T x \).
  (\( c^T y \) incr. along segment, \( v_0 \in P_0 \)).

• w.l.o.g. \( P_0 \) polytope,

  \[
P_0 = \text{conv} (\text{vertices of } P_0).
  \]

  \( \Rightarrow \exists \text{ vertex } w \text{ with } \\
  c^T v_0 < c^T x \leq c^T w \)
**WHY?** Simple but powerful principle:

\[ x = \frac{1}{3} \sum_{v} \lambda v, \quad \sum_{v} \lambda v = 1 \]

where \( v \) are vertices of \( \text{Po} \)

\[ \Rightarrow c^T x = \sum \lambda w c^T w \quad \text{"weighted average"} \]

\[ \Rightarrow \text{some } c^T w \geq c^T x \]
by bijection of edge $e$ with $P_0$. 

$e$ must be bounded (b/c $c^{ty}$ increases along $e$, but objective bounded on $P$).
Thus ends at some vertex $V_1$,

$$c^T v_1 > c^T v_0 \quad \square.$$

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**Proof of N.V.C.**

Recall: if vertex $v_0$ given by

then

$$c^T x = m,$$

$$P_o = \{x \in P : c^T x = m - \varepsilon \}$$

for small $\varepsilon$. 
Assume rank $A = n$; else no vertices.

1. $P_0$ bounded.

Exercise: If $Q$ unbounded polyhedron, $x \in Q$, then $Q$ contains ray from $x$: 
$\{ x + \alpha y : \alpha \geq 0 \}$. 
Suppose $P_0$ unbounded, let $x_0 \in P_0$.

$\Rightarrow P_0$ contains ray $[x_0 + ay : a \geq 0]$ for some $y$. 
- as $P_0 \leq x_0 + c^+$, $y \in c^+$.

- Use ray to construct another minimizer $\nu$ contradicting uniqueness.

- By closedness:
  $\{V_0 + \alpha y : \alpha \geq 0\}$. 
but $c^T x$ constant along it.

2. The bijection:

$$\text{face } F \rightarrow V_0 \text{ of } P$$

$$F_0 := \{x : c^T x = m \}$

$$= F \cap P_0.$$ 

α: **Onto:** Every face $F_0$ of $P_0$ can be written this way for some $F$ of $P$. 

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**Note:** The image contains a diagram with a cube labeled $P$, a face $F$, and a vertex $V_0$. The bijection is illustrated by arrows connecting $F$ to $V_0$. The equation $F_0 := \{x : c^T x = m \}$ defines a face $F_0$ in terms of the quantity $c^T x$. The diagram also indicates that $F_0$ is the intersection of $F$ with $P_0$. The text states that every face $F_0$ of $P_0$ can be written in this way for some face $F$ of $P$. This is labeled as **Onto** in the diagram.
• Let $F_0$ nonempty face of $P_0$.

$$F_0 = \begin{cases} a_i^T x = b_i & i \in I \\ c^T x = m - \epsilon \\ a_j^T x \leq b_j & j \not\in I \end{cases}$$

Let

$$F_0 = \begin{cases} a_i^T x = b_i & i \in I \\ a_j^T x \leq b_j & j \not\in I \end{cases}$$

(remove middle equality)

• $F_0$ is a face by faces them, so just need to show $v_0 \in F_0$. 


Recall that $v_0$ was only vertex $v$ with $c^Tv \geq m-\epsilon$.

But $c^T \ast$ bounded above on $F$ implies $\Rightarrow$ reaches max $\geq m-\epsilon$ at vertex $v$ of $F$; thus $v = v_0$.

**6) Dimensions** (will also imply one-to-one).

Want to show $\dim F_0 = \dim F - 1$. 
• **Enough to show**

\[ F \subseteq \text{aff}(F_0 \cup \{v_0\}). \]

\[ \implies \dim F_0 \geq \dim F - 1. \]

(dim \(F_0 \leq \dim F - 1 \) bc \( F_0 = F_0 \) plane, \( F_0 \neq F \)).

**Cases:**

1. \( c^T x \leq m - \epsilon \),
2. \( c^T x > m - \epsilon \).

1. If \( c^T x \leq m - \epsilon \), segment \( x \to v_0 \) clearly hits \( F_0 \), thus \( x \in \text{aff}(F_0 \cup \{v_0\}) \).
Else, \( x \) is in polyhedron 
\[
F' = F \cap \{ x \mid c^T x \geq m - \epsilon \}
\]
- \( F' \) is bounded (for same reason as \( P_0 \)).
  
\[ \Rightarrow \] \( F' \) convex hull of its vertices.

- Vertices of \( F' \) are all either
  a) on \( c^T x = m - \epsilon \) or
b) equal to $V_0$. 

(b/c they are vertices $v$ of $F$ satisfying $c^T v \geq m-\varepsilon$, $V_0$ only such vertex).

\[ \Rightarrow F' \subseteq \text{conv}(F_0 \cup \{V_0\}). \]

\[ \subseteq \text{aff}(F_0 \cup \{V_0\}). \]