Lecture 8

Plan:

- Faces of Polyhedra
- State facts of facts
- Prove them
Faces of Polyhedra

**Def:** \( a^{(1)} \ldots a^{(k)} \in \mathbb{R}^n \) are affinely independent if

\[
\sum_{i=1}^{k} \lambda_i a^{(i)} = 0
\]

and \( \sum_{i=1}^{3} \lambda_i = 0 \) imply \( \lambda_1 = \ldots = \lambda_k = 0 \).

(\( \sum_{i=1}^{3} \lambda_i = 0 \), is just linear indep.)

Linear independence \( \Rightarrow \) affine independence.

**Note:** \( \text{aff}(X) = \) lowest dim. affine space containing \( X \).

\( \exists (i) \) \( a^{(i)} \) be independent iff
La ${a_1, \ldots, a_k}$ affinely independent

\[
\begin{bmatrix}
    a_1 \\
    1
\end{bmatrix}
\]

linearly independent.

$\iff \text{aff}(\{a_1, \ldots, a_k\})$ has dimension $k-1$

Picture:

affinely independent in $\mathbb{R}^2$

affinely dependent in $\mathbb{R}^2$.

**Def.** Dimension $\dim(P)$ of polyhedron $P$:
\[-1 + \max \, \# \text{ affinely independent points in } P.\]

Equivalently, dimension of affine hull \( \text{aff} \, P \).

**Examples:**

- \( P = \emptyset \), \( \dim(P) = -1 \)
- \( P = \) singleton, \( \dim(P) = 0 \)
- \( P = \) line segment, \( \dim(P) = 1 \)
\[ \text{aff}(P) = \mathbb{R}^n \]

\[ \dim(P) = n; \]
\[ P \text{ "full dimensional"} \]

E.g., cube in \( \mathbb{R}^3 \):
\[ \{ x : 0 \leq x_i \leq 1 \} \]

\[ \dim P = 3 \]
\[ \dim \mathbb{R}^3 = 3 \]

(As polyhedron)

Why affine, not linear? Affine independence is translation invariant.

\[ \text{if I used max # lin indep points} - 1 \]
\[ \dim(P) = 1 \]
\[ \dim(P') = 0 \]
\[
1 = \dim(P) = \dim(P').
\]

**Def:** \(\alpha^T x \leq \beta\) is a valid inequality for \(P\) if \(\alpha^T x \leq \beta\) for all \(x \in P\).

**Def:** A face of a polyhedron \(P\) is \(\{x \in P : \alpha^T x = \beta\}\) for
\( \alpha^T x \leq \beta \) valid.

**Faces:**

- Faces are polyhedra
- Empty face \& entire \( P \) are called trivial faces
- dim = 1
- else \( F \) nontrivial

\[ 0 \leq \dim(F) \leq \dim(P) - 1 \]

- \( F : \dim(F) = \dim(P) - 1 \) called facets.
- \( v : \dim(F) = 0 \) called vertices.
Ex: list the 28 faces of the cube

\[ P = \{ x \in \mathbb{R}^3 : 0 \leq x_i \leq 13 \} \]

Fact: \( \infty \) many valid ineqs, but \( \# \) faces finite!
EVERYTHING
ABOUT POLYHEDRA

\[ A = \begin{bmatrix} -a_1^T & \cdots & -a_m^T \end{bmatrix} \quad P = \{ x : A x \leq b \} \subseteq \mathbb{R}^n \]

**Face Characterization:**

Any nonempty face of \( P \) is

\[ F_I = \left\{ x : \begin{array}{l} a_i^T x = b_i \quad \forall i \in I \\ a_i^T x \leq b_i \quad \forall i \notin I \end{array} \right\} \]

for some set \( I \subseteq \{1, \ldots, m\} \).

(and \( F_I \) is always a face)
Facet Maximaliy: The facets are the maximal nontrivial faces of a nonempty polyhedron $P$. For vertices: just need equalities. Vertices = extreme points. Exercise
**Vertex Characterization:**

Suppose $x^*$ extreme point of $P$

Then $\exists I$ s.t. $x^*$ is the unique soln to

$$\alpha_i^T x = b_i \text{ for all } i \in I$$

Moreover any $x \in P$ that uniquely solves $x^*$ is extreme.

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e.g. simplex $(0,1,0)$ is intersection of 3 constraints
• **Vertex minimality**: For \( \text{rank}(A) = n \), minimal nontrivial faces of polyhedron \( P \) are the vertices.

  Exercise: if \( \text{rank}(A) < n \), no vertices?

• **Polytopes = convex hulls**

  If a polyhedron \( P \) is bounded then \( P = \text{conv}\{\text{extreme points of } P\} \).

  (special case of Krein-Milman theorem: compact convex subset of \( \mathbb{R}^n \) is \( \text{conv}(\text{extreme pts}) \)).

• **Facets Characterize**
\[ \text{equality } a_i^T x \leq b_i \text{ is redundant if } P \text{ unchanged when it's removed} \]

\[ I_\neq := \{ i : a_i^T x = b, \forall x \in P \} \]

\[ I_\leq := \{ i : \exists x \in P \ a_i^T x < b \} \]

**Then:**

**Sufficiency:** If face \( a_i^T x \leq b \) for \( i \in I_\leq \)

is not facet, then \( a_i^T x \leq b \) is redundant.

**Necessity:** If \( F \) is facet of \( P \), \( \exists i \in I_\neq \) such that \( F \) is induced by \( a_i^T x = b_i \).
Near vertices = cones over polytopes

Let $v_0$ vertex of $P$ from valid inequality $c^T x \leq m$. Let $\varepsilon$ be such that $c^T v' \leq m - 3$
for all other vertices \( v' \).

Then

\[
P_0 = \{ x \in P : \sum x = m - 3 \}
\]

is a polytope & is bijection

\[
\{ P_0 's \ dim \ k \ faces \}
\]

\[
\leftrightarrow
\]

\[
\{ P's \ dim \ k+1 \ faces \ contain \ V_0 \}
\]

- **P's "graph" connected**: Graph of vertices & edges of polyhedron \( P \)
is always connected.

In particular: if $v^*$ minimum of $c^T x$ over $P$, $v_0$ vertex, $\exists v_0 \rightarrow v^*$ path which decreases objective.

$v_0 \hat{1} \geq 0$

bijective: $\dim k$ face $F$ of $P_0$ $\rightarrow$ $\dim k+1$ face of $P_0$ $P$ containing $F$ and $v_0$. 
Recall face characterization:

Let \( A \in \mathbb{R}^{m \times n} \), 
\[
A = \begin{bmatrix} - & \cdots & - \\ \vdots & \ddots & \vdots \\ - & \cdots & - \end{bmatrix}
\]

Any nonempty face of \( P = \{ x : Ax \leq b \} \)

is

\[
\left\{ x : \begin{array}{l}
a_i^T x = b_i \quad \forall i \in I \\
a_i^T x \leq b_i \quad \forall i \notin I
\end{array} \right\}
\]

for some set \( I \subseteq \{1, \ldots, m\} \).

Proof of converse: exercise
Proof

Consider valid inequality

\[ a^T x \leq b \]

giving nonempty face \( F \).

\[ F = \{ x : a^T x = b \} \cap P \]

\[ \text{max } a^T x \]

\( (P) \) subject to \( A x \leq b \)

\[ \begin{align*}
\cdot \text{ Let } y^* \text{ optimal solution to dual.} \\
& y^* = (y_1, \ldots, y_m) \\
\cdot \text{ Complementary slackness:} \\
& \text{optimal solutions } F \text{ are} \\
& \{ x : a_i^T x = b_i \text{ for } i : y^*_i > 0 \} \}
\]
Thus we can take \( I = \{ i : y_i^+ > 0 \} \). 

\[ \text{Ex:} \quad \begin{align*} &\text{positive orthant } \{ x \in \mathbb{R}^n : x_i \geq 0 \} = \emptyset \text{ has } 2^n + 2 \text{ faces} \quad \text{in inequalities} \\
&\text{How many of dim } \mathbb{R}^k? \quad \{ x_i = 0 : i \in I \} \end{align*} \]

For polytopes can also bound \# faces in terms of \# vertices. ("upper bound theorem" "Dehn Somerville equations")
pf: Exercise to prove from face characterization.

Recall vertex characterization:

Let $x^*$ extreme point for $P$.

Then $\exists I$ s.t. $x^*$ is the unique soln to

\[ a_i^T x = b_i \quad \forall i \in I. \]

Moreover, any such unique solution $x^* \in P$ is extreme.
**Proof:** Given extreme point $x^*$,

- define $I = \{ i : a_i^T x^* = b_i \}$.
- Note for $i \notin I$, $a_i^T x^* < b_i$.
- By "feasibility characterization", $x^*$ uniquely defined by

$$F = \begin{cases} 
(\star) & a_i^T x = b_i \quad \text{if } i \in I \\
(\star\star) & a_i^T x \leq b_i \quad \text{if } i \notin I. 
\end{cases}$$

- Suppose 3 other soln. $x$ to $(\star)$ (for contradiction).
- Because $a_i^T x \leq b_i$ for $i \notin I$,
\[
\hat{x}^3 + x^*(3-1)
\]

Still satisfies (*) \((**\) for \(\varepsilon\) small enough.

- Contradicts \(F\) having only one point. \(\Box\).

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**Basic Feasible Solutions:**

For \(Q = \{Ax=b, x \geq 0\}\) can describe extreme points very explicitly.

(every \(P\) can be put in this form).
Corollary of Vertex Thm: Extreme pts. of \(Q\) as above come from setting
\[x_j = 0 \text{ for } j \in J\]
and finding unique solution to \(Ax = b\) for remaining variables.

Can say more: Extreme points of \(Q\) as above are the basic feasible solutions (bfs), feasible solns obtained as follows:

FILLED IN LEC 7 HANDOUT

- Remove redundant rows from \(A\) ( )
Choose \( m \) columns \( B \) of \( A \). (\( \sim \))

\[
\begin{array}{c|c}
A & \begin{bmatrix} b \\
\end{bmatrix} \\
\end{array}
\]

Solve \( A_B \mathbf{x}_B = \mathbf{0} \), set

\[
x^*_i = \begin{cases} 
1 & \text{if } i \in B \\
0 & \text{else}
\end{cases}
\]
Recall vertex minimality

If $\text{rank } A = n$, vertices are minimal nontrivial facets of $P$.

$$P = \{ x : A x \leq b \}$$

**Proof:** Let $F$ min'l face of $P$.

- **Face characterization** $\Rightarrow \exists I$
\[ F = F_I = \begin{cases} x : \begin{align*} &a_i^T x - y_i + \epsilon \\ &a_j^T x \leq b_i ; \forall i \neq I \end{align*} \end{cases} \]

Assume no redundant inequalities and adding any elt to I makes \( F_I \) empty. (We else \( F \) is empty). (We else \( F \) face \( \neq F \)).

- **Consider two cases:**
  - **Only the equalities are needed** (a\( j^T x \leq b_j \) redund. for F).
  - i.e. \( F \) is exactly
    \[ \{ x : a_i^T x = b_i ; \forall i \in I \} \].

- **Claim:** \( \forall j \neq I, \ a_j \in \text{lin}(a_i : i \in I) \).

- **Else** \( a_j^T x = b_j + 1 \) has solution in \( F \), contradicting \( a_j^T x \leq b_j \).
\((a_i^\top x \mid i \in I) \text{ do not determine } a_j^\top x \text{ unless } a_j \in \text{im}(a_i : i \in I)\)

\[\text{Equivalently: Submatrix } A_I \text{ w/ rows in } I \text{ satisfies}
\]
\[\text{row}(A) = \text{row}(A_I)\]

\[\begin{array}{c|c}
  & n \\
  \hline 
  i & \\
  \hline 
  j & \\
  \hline 
  A_I & A \\
  \hline 
\end{array}\]

\[\text{rank}(A_I) = \text{rank}(A) = n.\]

\[\text{Thus: } a_i^\top x = b_i \text{ for } i \in I\]

has unique soln, so \(F\) is single point, i.e. a vertex. \(
\checkmark\)
(b) Some inequality needed; we'll show is contradiction.

- \( \exists j \notin I, \tilde{x} \text{ w/ } (\tilde{x} \notin P) \)
  \[
  a_i^T \tilde{x} = b_i \text{ i.e } I,
  
  a_j^T \tilde{x} > b_j
  \]

- \( F \text{ nontrivial } \Rightarrow \exists x \in F. \)
  \( \tilde{x} \text{ satisfies } \)
  \[
  a_i^T x = b_i \text{ i.e } I,
  
  a_j^T \tilde{x} \leq b_j
  \]
Consider convex combination

\[ x' = \lambda \tilde{x} + (1-\lambda) \tilde{x} \]

Let \( \lambda \) be largest so \( x' \in P \).

Finally, we can show equiv b/w bounded polyhedra & convex hulls (polytopes).

Finally we can show equiv b/w bounded polyhedra & convex hulls (polytopes).
Recall: $P = \{ A x \leq b \}$ bounded
then $P = \text{conv}(\text{x-treme pts. of } P)$.

i.e. $P$ is polytope.

Proof: Use TOTA

- $x \in P \Rightarrow \text{conv}(x) \subseteq P$.

- Assume for contradiction that $\text{conv}(x) \not\subseteq P$.

- Let $x' \in P \setminus \text{conv}(x)$.

- Then

$$\sum_{u \in x} \lambda_u x = x$$

$$\sum_{v \in x} \lambda_v = 1$$
\[ \lambda v \geq 0 \]

has no solution.

\[ \overrightarrow{\text{TOTA}} \Rightarrow \]

\[ \tilde{A} \rightarrow \begin{bmatrix} \vdots & \sqrt{\lambda} & \vdots \\ \vdots & 1 & \vdots \\ \vdots & \vdots & \vdots \\ -I \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \]

\[ \leq \]

\[ \Delta \rightarrow \square \]

\( \leq \rightarrow \geq \)

\( \geq \rightarrow ? \)

has no soln \( \Rightarrow \)

\[ A^T y = 0, \quad b^T y < 0, \quad y \geq 0 \]

has soln. i.e.

\[ \overrightarrow{A^T} \quad y \]

\[ 1 \quad 1 \]
for $s \geq 0$, and $y^T b < 0$.

**I.E.**

\[ (*) \quad t + c \cdot v \geq 0 \]

\[ (**) \quad t + c \cdot x < 0. \]

- $P$ bounded $\Rightarrow$
  \[ \min \{ c^T x : x \in P \} = z^* > -\infty. \]

\[ \forall v \in X \Rightarrow c \cdot x < c \cdot v \forall v \in X \]
Face induced by $\{x \mid \mathbf{c}^T x = z^*\}$ is nonempty, but contains no vertex.

(because $\emptyset \Rightarrow$ objective is less on $x$ than any vertex.)

Contradicts vertex minimality! [Orig \checkmark A \downarrow \checkmark]

(appplies b/c rank $A = n$; if rank $A < n$, $P$ not bounded (b/c some solution to $Ay = 0$). □
assume w.l.o.g. $0 \leq P \Rightarrow b \geq 0$