Lecture plan:

1. non-bipartite matchings.
2. Tutte-Berge
3. Algorithmic proof (Edmonds' algo)

* might not finish!
Non-bipartite Matching

- Given $G = (V, E)$, do not assume bipartite.
- Want maximum matching $M$ in $G$.
- König's theorem doesn't hold:
  \[ \text{max matching} \leq \text{min vertex cover}. \]

- Recall from lecture 1: instead, duality w/ obstructions based on parity.
**Tutte-Berge Formula**

Given $U \subseteq V$, 

**Def.** 

$G \setminus U := G$ after deleting $U$ & all adjacent edges.

$\omega(G \setminus U) := \# \text{ odd connected components in } G \setminus U$ 

(\# \text{ c.c.'s w/ odd } \# \text{ of verts}).

**E.g.** 

$G = \begin{array}{c}
\text{Graph}
\end{array}$ 

$G \setminus U = \begin{array}{c}
\text{Graph}
\end{array}$

$\omega(G \setminus U) = \omega(\begin{array}{c}
\text{Graph}
\end{array}) = 3$
**Theorem (Tutte-Berge Formula):**

$$\max |M| = \min \frac{1}{2} (|V| + |U| - o(G(U)))$$

**Proof (\(\leq\)):** i.e., "weak duality"

- Deleting \(U\) deletes \(|U|\) edges of \(M\).

How many left over?
Here, the leftover is at most
$$\sum_{i=1}^{3} \left\lfloor \frac{|K_i|}{2} \right\rfloor$$

Thus, if $K_1, \ldots, K_k$ are connected components of $G \setminus U$,

* $|M| \leq |U| + \sum_{i=1}^{k} \left\lfloor \frac{|K_i|}{2} \right\rfloor$.

Can rewrite:

$$\left\lfloor \frac{|K_i|}{2} \right\rfloor = \begin{cases} |K_i| & \text{if } |K_i| \text{ even} \\ \frac{|K_i|-1}{2} & \text{else.} \end{cases}$$

Thus

$$\sum_{i=1}^{k} \left\lfloor \frac{|K_i|}{2} \right\rfloor = \sum_{i=1}^{k} \frac{|K_i|}{2} - \frac{1}{2} o(G \setminus U) = \frac{|U| - |M| - o(G \setminus U)}{2}$$
**Proof of $\geq$ ?**

- Beautiful algorithm due to Edmonds.
- Challenge: though still true that $M_{\text{max}} \iff \text{no augmenting path w.r.t } M$.

Finding the paths is hard.

- Plugging $\star \star$ into $\star$ gives

$$|M| \leq \frac{|V|}{2} + \frac{|U|}{2} - o(G \setminus U)$$

E.g.

$$G = \begin{array}{c}
\begin{array}{c}
\hline
1 & 2 & 3 & 4 \\
\hline
5 & 6 & 7 & 8 \\
\hline
9 & 10 & 11 & 12 \\
\hline
13 & 14 & 15 & 16 \\
\hline
\end{array}
\end{array}$$

$$|M| = 4, \quad \frac{1}{2}(|V| + |U| - o(G \setminus U))$$

$$= \frac{1}{2}(9 + 2 - 3) = 4.$$
- Why? Natural approach repeats vertices. *(Fails)*

Natural approach: whenever you see $u \rightarrow v \in E$, add directed edge $uv$:

\[ u \rightarrow v \in E \]

**E.g.**

Then, start at exposed vertex & look for vertex adjacent to an exposed vertex in blue digraph.

Problem: can lead to repeated vertices.
when we first repeat, we have found a flower (with respect to M):

**Stem**: even-length alternating path from exposed vertex $u$ to vertex $v$

**Blossom**: odd-length cycle intersects stem in $v$ alternately except for edges incident to $v$. 

E.g.
Algorithm idea:

At each step, have matching $M$.

- find augmenting path or blossom
  w.r.t. $M$ or show neither exists.
- If neither exists, matching is maximum. (b/c no augmenting path)
  
- if augmenting path, augment & repeat.
• If flower, let B be blossom. Create graph G/B (not G\B).

Called contraction where:

1. B shrunk to single vert. b
2. Edges (u,v) u \notin B, v \in B replaced by (u, b) \in G/B.

E.g.

G → G/B

G

M/B

M

a

b
Note: is matching $M/B$ in $G/B$ and
\[ |M| - |M/B| = \frac{|B|-1}{2} \]
(i.e. # edges of $M$ in $B$).

**Crucial Theorem:** Let $B$ be a blossom w.r.t. $M$. Then

\[
\begin{align*}
M \text{ max matching in } G \\
\iff \\
M/B \text{ max matching in } G/B.
\end{align*}
\]

Proof will be algorithmic:
If bigger matching in $G/B$ then
M/B, can use it to find bigger matching in $G$ than $M$.

**Theorem $\rightarrow$ Algorithm:** recursion!

Assuming we can find either any path or blossom, can recurse to increase size of $M/B$ in $G/B$.

* if not possible: $M$ maximum.
* else: use new matching in $G/B$ to increase $M$

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**Proof of Crucial Theorem:**

(1) **W.L.O.G. assumption:**
If $P$ nonempty, look at $M\Delta P$.

- $M\Delta P$ has empty stem & blossom $B$. 

why w.l.o.g.?
Proving theorem for MΔP also proves for M:

\[ M \text{ maximum in } G \]

\( P \text{ alternating } M = (M\Delta P) \)

\[ \iff \quad M\Delta P \text{ maximum in } G \]

Theorem for empty stem

\[ \iff \quad M\Delta P/B \text{ max in } G/B \]

\[ M\Delta P/B = (N/\beta)\Delta P \]

\[ \iff \quad (M/B)\Delta P \text{ max in } G/B \]

\( P \) still alternating.

\[ \iff \quad (M/B) \text{ max in } G/B. \]
Finally, start proof of crucial claim:

\[ M \max, \text{in } G \]

\[ \iff M/B \max \text{ in } G/B \]
1. (⇒): 

Contra: Suppose M/B not max, show M not max.

Suppose N is matching in G/B larger than M/B.

- Pull back N to matching \( \tilde{N} \) in \( G : \tilde{N}/B = N \). \( \tilde{N} \) incident to \( \leq 1 \) vertex of B.

- Expand to matching \( N^* \) in \( G^* \): add \( \frac{1}{2}(|B|-1) \)
edges in $B$.

$|N^+|$ exceeds $|M|$ by same amt. $|N|$ exceeds $|M/B|$. 
Suppose $M$ not max in $G$.

- Then I aug path $P$ between exposed verts $u, v \in G$.
- w log $u \& B$, $B$ has only 1 exposed vertex. (stemisempty).

$w := \begin{cases} 
\text{first vertex of } P \text{ in } B 
& \text{(starting from } u) 

v & \text{if } P, B \text{ share no vertices.} 
\end{cases}$

- $Q := \text{part of } P \text{ between } u, w$.
- $Q \text{ augments path in } M/B$
b exposed in M/B bc stem is empty.

Also: if P, B vertex disjoint, then P m/B.

Finally: Augmenting M/B along Q

⇒ M/B not maximum
\[ \frac{\hat{M}}{B} = M^* \]

\[ B \text{ is a blossom for } M \text{ maybe not for } \hat{M}. \]

\text{Subtlety: } \text{This doesn't say maximum matching } m^* \text{ in } G/B
max matching $\mathcal{M}$ in $G$ by adding $\frac{|B|-1}{2}$ edges from $B$ to $M^*$!

Ex. find example of above!

i.e. Blossom $B$ of $M$ so that $M^* \text{ max in } G \setminus B$ but adding the $\frac{|B|-1}{2}$ edges to $M^*$ in $G$ doesn't result in max matching $\hat{M}$ in $G$.

explain why no contradiction.
Lecture 5  Plan:
1. Finish Edmonds’ alg.
2. Prove Tutte-Berge
3. (Maybe) Start polyhedra.

Announcements:
- I may be 10-15 mins late Thurs. (will keep you posted on slack).
- HW due 11:00pm Thurs.

Edmonds’ Algorithm
(Given M, find augmenting path/flower)
- Label exposed vertices EVEN:
Keep others unlabelled initially. (eventually will label some **EVEN*/ODD*)

- Maintain alternating forest:
  - (sub) graph in which each connected component is alternating tree (AT)
  - i.e. tree st. paths to root are
    - (i) alternately w.r.t. $M$
    - (ii) alternate b/w odd & even child
    - (iii) leaves are even.

- Process **EVEN** vertices one at a time.
Currently on \( u \in \text{EVEN} \).

All verts. in a tree are exposed or matched within the tree.

\( \text{Case 1:} \) If edge \((u,v)\) with \( v \) unlabelled, label \( v \in \text{ODD} \). \( v \) not exposed.

\( \text{Case 2:} \) Else \( v \in \text{EVEN} \);

Label \( v \)'s mate \( w \in \text{EVEN} \).

Add
(b) If there exists an edge \((u,v)\) s.t. \(v\) even and \(v\) belongs to different AT than \(u\),

Then is an augmenting path between the roots.

have found no path; increase \(M\); start over with new \(M_0\).
If there is an edge \((u,v)\) with \(v\) labeled \text{EVEN} and \(v\) in \(u\)'s AT, then the paths \(u,v \rightarrow \text{root}\) form a flower with \((u,v)\).
Shrink to $G/B$.

- Keep same labelling & label $b$ \text{EVEN}.
- Recursively find max matching $m^*$ in $G/B$.

Using crucial theorems, can use $M^*$ to increase $M$.

Start over w/ new $M$.
- If none ($a$, $b$, $c$) apply, then $M$ is maximum.

**Correctness:** Suppose none of $a$, $b$, $c$ apply anymore for the \text{EVEN} vertices. \( u \).

**Recall:** $a) (u,v)$ v unlabelled
Claim: Current matching $M_k$ is max in current $G_k = (V_k, E_k)$.

Note: (i) unlabelled matched to eachother.

(ii) Odd vertices all matched to even vertices.

E.g.

Let's consider a specific example to illustrate this concept.
Proof of Claim: Consider $U = \text{ODD}$ and consider the upper bound from Tutte-Berge for $G_k$,

$$|M_k| \leq \frac{1}{2} \left( |V_k| + |U| - \delta(G_k \setminus U) \right)$$

- No edges between EVEN vertices, (else (b) or (c) applies).
- No edges between EVEN & unlabelled, (else (a) applies).

Thus, EVEN are singleton components in $G_k \setminus U$. 

$G_k := G / B_1 / B_2 \ldots / B_k$

$G_i \\
B_i$ blossom wrt $M_{i-1}$ in $G_{i-1}$
so \( o\left(G_k \setminus \text{ODD}\right) \geq |\text{EVEN}| \) * 

- All **ODDS** matched to **Evens**
- All **unlabeled** matched to **unlabeled**, so

\[
\begin{aligned}
|M_k| &= |\text{ODD}| + \frac{1}{2}(|V_k| - |\text{ODD}| - |\text{EVEN}|) \\
&= \frac{1}{2}(|V_k| + |\text{ODD}| - |\text{EVEN}|)
\end{aligned}
\]

**plug** * into **:**

\[
|M_k| = \frac{1}{2}(|V_k| + |\text{ODD}| - |\text{EVEN}|)
\]

\[
\geq \frac{1}{2}(|V_k| + |\text{ODD}| - o(G_k \setminus \text{ODD}))
\]

Tutte-Berge (upper bound) \( \Rightarrow \) 

\( M_k \) max in \( G_k \) claim proven.
Apply crucial theorem repeatedly for $B_k B_k^{-1} \cdots B_1$.

Shows: algorithm constructs a max matching in $G$ because $B_j$ blossom for $G_{i-1}$ for $M_{i-1}$.
Running time.

- Algorithm performs augmentations of matching ("outer loop")
- Between two augmentations ("inner loop") shrinks blossom
  \( \leq \frac{n}{2} \) times (shrinking removes \( \geq 2 \) vertices).
- Time to construct \( AF \) is \( O(m) \), \( m := |E| \).
So overall, \( O(n^2m) \).

**Proof of Tutte-Berge (≥)**

We showed: TB holds for graph \( G_k \) for which alg. terminates.

- Recall
  
  \[ G_i : G/B_{i-1} \]

  \[ M_i : M/B_{i-1} \]

  \[ G_0 : G \].

- TB holds for \( G_k \), i.e.

  \[ M_1 \geq \frac{1}{2}(|V|+|U|) - o(G/u) \]
\[ |M_k| = \frac{1}{2}(|v_k| + |u| - o(G_i/u)) \]

where \( U = \text{ODD} \),

\( \text{blc } G_k \backslash \text{ODD} = \text{EVEN} \);

even singleton \( \text{Cc} \).

- Unshrink \( B_i \) one by one.

**Claim:** \( U = \text{ODD} \) obtains equality in \( TB \) for all \( G_i, M_i \).

\( M_i = \frac{1}{2}(|v_i| + |u| - o(G_i/u)) \).
backwards induction.

In step $G_i \rightarrow G_{i-1}$:

(i) $|V_{i-1}| = |V_i| + |B_i| - 1$

and $b_i$ itself
\[ |M_{i-1}| = |M_i| + \frac{1}{2}(|B_i| - 1) \]

(ii) Unshrinking \( B_i \) adds even \( |(B_i| - 1) \) vertices to some C.C. of \( G_i \setminus U \), so \# odd/even components stays same.

\[ o(G_{i-1} \setminus U) = o(G_i \setminus U) \]

(iii) Using this, when \( i < i-1 \)
the RHS & LHS of

\[ \text{max} |M1| = \frac{1}{2} \left( |V1| + |\mathrm{U1}| - o(G \setminus \mathrm{U}) \right) \]

increase by \( \frac{1}{2}(1\text{Bi1}-1) \).

Apply induction to conclude \( \square \).

\( \text{TB: } \)

\[ \text{max} |M1| \leq \min \frac{1}{2} \left( |V1| + |\mathrm{U1}| - o(G \setminus \mathrm{U}) \right) \]

(i) \( |M1| \leq \frac{1}{2}(|V1| + |\mathrm{U1}| - o(G \setminus \mathrm{U}) \)

(ii) here: \( \text{for some } M \)

\[ |M1| = \frac{1}{2}(|V1| + |\mathrm{U1}| - o(G \setminus \mathrm{U}) \). \]
Corollary of Tutte-Boye: 

\[ G \text{ has p.m. i.f.f. } \exists u, v \in V(G) \text{ s.t. } \delta(G - uv) < |M|. \]

This is called Tutte's matching theorem.