Lecture 15

Plan

1) Finish min T-odd cut (see Lec(4 notes)

2) Matroids.

• Set 4 deadline extended to Mon Apr 26

• No OH upcoming Monday evening
Matroids

"Tractable" set systems.

E.g. Given vectors $v_1, \ldots, v_m \in \mathbb{R}^n$, consider set system $I \subseteq 2^{[m]}$:

$$I = \{ S \subseteq [m] : \forall v_i : i \in S \text{ linearly independent} \}$$

picture:

Not in $I$!

"dependent".

$x$ also $\emptyset \in I$.
Properties of $I$:

(P1) "Downward closed"
If $X \subseteq Y$ and $Y \in I$, then $X \in I$.

(P2) "Exchange property"
If $X \in I$ and $Y \in I$ and $|Y| > |X|$, then we can add $Y$ to $X$ while maintaining independence of $X$.

Formally: $\exists e \in Y \setminus X$ s.t. $X \cup \{e\} \in I$. $x + e$
**Proof of P2:**
\[ |x| = \text{dim} \text{span} \{v_i : i \in X \} \]
\[ |y| = \text{dim} \text{span} \{v_i : i \in Y \}. \]
\[ \Rightarrow \text{span} \{v_i : i \in X \} \subseteq \text{span} \{v_i : i \in Y \}. \]
\[ \Rightarrow \exists j \in Y \text{ s.t. } v_j \notin \text{span} \{v_i : i \in X \}. \]
\[ \Rightarrow x+j \in I. \qed \]

**P1, P2 capture combinatorial structure of I.**

**For matroids:** take P1, P2 as axioms.

\( I \subset 2^E, \) \( x \) is maximal in \( I \) if \( \exists \) no \( Y \in I \) s.t. \( X \subseteq Y. \)

**Def (Matroid):** A matroid \( M \) is a pair \( (E, I) \) where
- \( E = E(M) \) finite set called **groundset** of \( M. \)
- \( I = I(M) \subseteq 2^E \) called **independent sets.**
I satisfies P1 & P2.

**Remarks**
- P2 ⇒ all maximal (by inclusion).
- Independent sets have same size.
  - (else could increase by P2).
- Maximal independent set called a base of M.
  - Dependent := not independent
  - for $F \subseteq E$, the restriction $M|_F$ is another matroid.

**Examples**
- Linear matroid: example from beginning.

**Def** equivalent: $A \in \mathbb{R}^{n \times m}$ matrix,
I = \{ \text{subsets of cols. of } A \text{ s.t. submatrix } A_S \text{ has rank } A_S = |S| \} \\
E = \{ \text{set of columns of } A_S \}. \quad |E| = n \\
e.g. \\
A = \\
\begin{bmatrix}
A_S \\
\end{bmatrix} \\
\text{write } M = M_A \text{ if } M \text{ comes from } A. \\
\text{This makes sense for } A \in F^{m \times n} \text{ for any field } F. \\
\text{bases of } M_A: \text{subset } S \text{ s.t. cols. of } A_S \text{ are a basis for } \mathbb{R}^n. \\
\text{“boring” example: uniform matroid: } U_{n,K} = (E,I) \\
\text{where } |E| = n \\
U_{n,2} \text{ complete graph on } I = \{ \text{all subsets of } E \text{ of size } \leq K \} \\
= \{ S \subseteq E: |S| \leq K \}. 
a vertices.

\[ \text{the free matroid is } U_n, n = 2^E. \]

- Partition matroid: \( M = (E, I) \)
  where \( E \) is disjoint union \( E_1 \cup \ldots \cup E_k \)

\[ I = \{ X \subseteq E : |X \cap E_i| \leq k_i \} \]

for fixed \( k_1, \ldots, k_k \) (parameters).

\[ \text{e.g.} \]

\[ \begin{array}{ccc}
E_1 & E_2 & E_3 \\
k_1 & k_2 & k_3
\end{array} \]

Check P2:

- Let \( |X| < |Y| \), \( X, Y \in I \).
- \( \exists i \text{ s.t. } |Y \cap E_i| > |X \cap E_i| \)
• If for any $e \in Y \setminus E_i \setminus X \setminus E_i$ $x + e$ independent.

**Remark:** if $E_i$ not disjoint:

\[ k_i = 1 \]

\[ E_1 \quad E_2 \]

\[ X \quad T \]

\[ k_L = 1 \quad \text{fails } P_2 \]

• Another Nonexample: set of matchings in a graph.

\[ \text{e.g.} \]

\[ Y \quad Z \quad X \quad \text{fails } P_2 \]
graphic matroids:

Given graph $G = (V, E)$, undirected.
Let $M(G) = (E, I)$ where

$I = \{ \text{forests in } G \}$

$= \{ \text{acyclic subgraphs of } G \}$.

\[ G = \triangle \]

\[ I = \{ \triangle, \triangle, \triangle, \triangle, \ldots \} \]
Checking P2:

- If forest \( F \) \( \Rightarrow \ |V| - |F| = \) components
- \( x, y \) forests, \( |x| < |y| \) \( \Rightarrow \) \( y \) has fewer connected components.

\( \Rightarrow \) some edge \( e \) of \( Y \) connects two connected components of \( X \).

\( \Rightarrow X + e \) larger forest.

D bases: the spanning trees. (all have \( n-1 \) edges).

D graphic \( \Rightarrow \) linear.
PF: \( M(G) = MA \) where \( A \) directed vertex-edge incidence matrix (direct arbitrarily).

\[
E(i,j) \\
A = \begin{bmatrix}
\vdots & \vdots & & \vdots \\
1 & & & -1 \\
\vdots & \vdots & & \vdots \\
\end{bmatrix}
\]

Ex. Check: subset of cols. is lin. indep \( \iff \) subgraph contains no cycle. \( \square \)

Graphic \( \Rightarrow \) regular:

Say \( M \) regular if \( M \) is linear over every field \( F \).

\(-1 \text{ in } A \iff \text{ additive inverse of } 1 \text{ in } F.\)

Note: \( A \) above is T.U.

Fact: matroid \( M \) regular \( \iff \) 
\( M = MA \) (over \( F \)) for T.U. matrix \( A \).
Circuits

- circuit: = minimal dependent set.
  (i.e. \( L \text{ circuit} \Rightarrow C - e \text{ independent} \))

  e.g. in graphic matroid: circuits are the cycles.

  in partition matroid, circuits are just subsets \( C \subseteq E_i \)
  with \( |C| = k+1 \).

  e.g. 

  \[
  C \quad \begin{array}{ccc}
  \text{E}_1 & \text{E}_2 & \text{E}_3 \\
  \text{k}_1 & \text{k}_2 & \text{k}_3
  \end{array}
  \]

  Note: 

  \( L \text{ circuit} \Rightarrow C - e \text{ independent} \)
There’s exactly one way to do the reverse:

**Theorem** (unique circuit property)

\[ \text{Let } M = (E, I) \text{ matroid.} \]
\[ \text{Let } S \subseteq I, \text{ e } \in E \text{ s.t. } S + e \not\subseteq I \]
\[ \text{Then: } \exists \text{ a unique circuit } C \subseteq S + e. \]

E.g. If F is a forest, F + e isn’t:

![Diagram of a forest F and a circuit C with edge e added]

**Remark:** Uniqueness shows
How to make more independent sets: Let $C \subseteq S + e$ circuit, $f \in C$ then $S + e - f \in I$.

**Proof of UCP:**

- Suppose $S + e$ contains distinct circuits $C_1 \neq C_2$. was hypo class.
- Minimality $\Rightarrow C_1 \notin C_2 \Rightarrow \exists f \in C_2 \setminus C_1$

Note: $C_2 \subseteq S + e - f$. 

E.g. $C \rightarrow C'$
We'll show $S \cup \{f\} \cup I$, contradicts $C_2 \subseteq S \cup \{e\}$.

- $C_1 \cap f$ independent $\Rightarrow C_1 \cap f$ to maximal indep. $X$ subject to $X \subseteq S \cup \{e\}$.

  e.g.

- Both $S \cup X$ maximal independent within $S \cup e$ $\Rightarrow |X| = |S| \text{ by } P2$.
\( e \in X \Rightarrow \text{ because } e \in C_1 \Rightarrow e \in X \Rightarrow X = S + e - f. \)

\( \Rightarrow S + e - f \text{ indep, contradiction. } \square \)

\( M = (E, I) \) could have \( E = \text{ edge set of } G \) (review P1, P2 from earlier). \( I = \text{ forests in } G. \)

**Matroid optimization**

- Given cost function \( c : E \to \mathbb{R} \), want indep. set \( S \) of max. cost

\[ c(S) = \sum_{e \in S} c(e). \]

- This problem is tractable, one reason matroids are important.
• if some $c(e) < 0$: can restrict to $M \setminus E - e$. (remove from $S \in I$ increases cost).

• if $c \geq 0$: need only optimize over bases.

e.g. for graphic matroids: on connected graphs this is the maximum spanning tree problem (MST).

Recall: MST has simple greedy algorithm: keep adding lowest element until forest creates a cycle.
Kruskal’s algorithm

- Fact: greedy algo works for any matroid.
- Actually, for all $K$: greedy outputs independent set of size $K$ of max cost. $S_K$

**Algorithm**

Let $|E| = m$.

1. Sort $E$ by cost: $c(e_1) \geq c(e_2) \ldots \geq c(e_e)$
2. $S_0 := \emptyset$, $K = 0$
For $j = 1$ to $m$:

- If $S_k + e_j \in I$ then:
  - $S_{k+1}' = S_k + e_j$
  - $k \leftarrow k + 1$

- Output $S_1, \ldots, S_k$

**Theorem:** For any matroid $M = (E, I)$, the above algorithm finds an independent set $S_k$ such that

$$c(S_k) = \max_{|S| = k} c(S).$$
Proof: Suppose not.

• Let \( S_k = \{d_1, \ldots, d_k\} \) with \( c(d_i) \geq \ldots \geq c(d_k) \).

• Suppose \( T_k = \{t_1, \ldots, t_k\} \)
  \( c(t_1) \geq c(t_2) \ldots \geq c(t_k) \) s.t. \( c(T_k) > c(S_k) \).

• Let \( p := \) first index where \( c(t_p) > c(d_p) \).

• Let \( A = \{t_1, \ldots, t_p\} \)
  \( B = S_{p-1} = \{d_1, \ldots, d_{p-1}\} \).
\[ |A| > |B| \implies \exists t_i \in A \setminus B \text{ st. } B + t_i \in I \text{ (by P2)}. \]

But \( c(t_i) \geq c(t_p) > c(\Delta p) \)
\[ \implies c(t_i) > c(\Delta p) \]
\[ \implies t_i \text{ should have been added to } S_{p-1} \text{ instead of } \Delta p. \]

To get global
\[ \max_{\text{min-cost independent set}} \]
In greedy alg,
Replace for \( j = 1, \ldots, m \)
by \( j = 1, \ldots, q \)

eq is last nonnegative element.