Instructions. This is a practice, take-home quiz. This is meant to be done in 2 hours with access to notes and course material, but no access to collaborators. For best practice I suggest trying to complete it under these conditions. Afterwards please tell me if 2 hours felt like enough.

1. Consider a bipartite graph $G = (V,E)$ in which every vertex has degree $k$ (a so-called $k$-regular bipartite graph). Prove that such a graph always has a perfect matching in two different ways:

   (a) by using König’s theorem,
   (b) by using the LP formulation of the min-weight perfect matching problem.

2. Suppose $G = (V,E)$ is a 2-edge-connected graph (that is, $G$ remains connected if you delete any single edge) with at least one perfect matching, and suppose that $G$ has a special edge $e$ such that the graph obtained by removing $e$ from $G$ has no perfect matching. Show that there is necessarily a nonempty set $S \subseteq V$ with the following properties:

   • the number of odd components of $G \setminus S$ is exactly $|S|$,  
   • $G \setminus S$ has at least one even component.

3. Show that for any point $x_0$ in an unbounded polyhedron $P \subset \mathbb{R}^n$, $P$ contains a ray from $x_0$, a set of the form $\{x_0 + \alpha y : \alpha \geq 0\}$ for some $y \in \mathbb{R}^n$. A suggested approach:

   • Show it is enough to prove this when $x_0$ is 0.
   • As $P$ unbounded, there is some $c$ such that $\max\{c^T x : x \in P\} = \infty$. Apply linear programming duality for the linear program $\max\{c^T x : x \in P\}$ to show something about the feasibility/infeasibility of the dual.
   • Apply Farkas’ lemma to the infeasibility/feasibility of the dual in order to obtain the direction of the ray.

An extra problem: This one shouldn’t count as part of your 2 hours, but a problem like it could appear on the exam.

4. Let $G$ be a bipartite graph with bipartition $A,B$ and edge set $E$. A fractional vertex cover is a pair of assignments of numbers $(x_a \in \mathbb{R} : a \in A)$ and $(y_b \in \mathbb{R} : b \in B)$ to the vertices such that

   \[
   x_a + y_b \geq 1 \quad \forall ab \in E \\
   x_a \geq 0 \quad \forall a \in A \\
   y_a \geq 0 \quad \forall b \in B
   \]

   The fractional vertex cover number is

   \[
   \tau(G) := \min \left\{ \sum_{a \in A} x_a + \sum_{b \in B} y_a : (x,y) \text{ is a fractional vertex cover of } G \right\}.
   \]
Show that the fractional vertex cover number is the same as the vertex cover number, i.e. the size of a minimum vertex cover.  \textbf{Hint:} \[1\]

\[1\]Use total modularity, and that $M^T$ is totally unimodular if and only if $M^T$ is.