Matroid intersection activity

Collaborate on these with your breakout room in explain.mit.edu (or using whatever method you find convenient).

1. Let $G$ be a bipartite graph with bipartition $A, B$.
   
   (a) Given an example showing that the set of matchings does not form the independent sets of a matroid.

   (b) Show that the set of matchings is the intersection of two matroids.  \( \text{Hint:} \) \(\uparrow\)

---

\(\uparrow\) It is the intersection of two partition matroids.
2. Recall a problem from Pset 4: given an undirected graph $G$ and an assignment $p$ of numbers to the vertices, we’d like to direct the edges in $G$ so that every vertex has at most $p(v)$ incoming edges.

(a) Describe a pair of matroids whose largest common independent set has size $|E|$ if and only if $G$ has a direction satisfying the above condition. **Hint:** ²

²Again, two partition matroids will suffice.
3. Suppose $G$ is an undirected graph and the edge set $E$ of $G$ has been “colored.” Show that the set of *colorful spanning trees* (spanning trees whose edges are all different colors) is the set of common bases of two matroids. **Hint:** This time you can use a graphic matroid and a partition matroid.
4. For a directed graph $D$ and a “root” vertex $r \in V(D)$ such that $r$ has no incoming edges, define an arborescence to be a spanning tree of $D$ directed away from $r$. 

(a) Let $G$ be the underlying undirected graph of $D$ obtained by forgetting the directions of the edges. Show that any subgraph of $D$ which corresponds to a spanning tree in $G$ and has at most one edge entering each vertex is an arborescence.

(b) Show that the set of arborescences of $D, r$ is the set of common bases of two matroids. **Hint:**

---

*Here spanning tree just means that it’s a spanning tree in the underlying undirected graph $G$.*

*G may have multi-edges if both directions $(u, v)$ and $(v, u)$ of an edge were present in $E(D)$.*

*Again you can use the intersection of a graphic matroid and a partition matroid.*
5. Consider an undirected graph $G$. We’d like to decide if $G$ is the union of two edge-disjoint spanning trees. Given a matroid $M = (E, I)$, define its dual matroid $M^*$ to be $(E, I^*)$ where $I^*$ is the set of subsets of $E$ whose complements contain a base of $M$.

(a) Describe a pair of matroids whose largest common independent set has size $|V| - 1$ if and only if $G$ has two edge-disjoint spanning trees. You may use that the dual matroid is indeed a matroid.

(b) **Bonus:** prove that the dual matroid is a matroid.