

Sidorenko's conjecture for a class of graphs: an exposition

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A famous conjecture of Sidorenko [2] and Erdős-Simonovits [3] states that if H is a bipartite graph then the random graph with edge density p has in expectation asymptotically the minimum number of copies of H over all graphs of the same order and edge density. The goal of this expository note is to give a short self-contained proof (suitable for teaching in class) of the conjecture if H has a vertex complete to all vertices in the other part. This was originally proved in [1].

Theorem 1 *Sidorenko's conjecture holds for every bipartite graph H which has a vertex complete to the other part.*

The original formulation of the conjecture by Sidorenko is in terms of graph homomorphisms. A homomorphism from a graph H to a graph G is a mapping $f : V(H) \rightarrow V(G)$ such that for each edge (u, v) of H , $(f(u), f(v))$ is an edge of G . Let $h_H(G)$ denote the number of homomorphisms from H to G . We also consider the normalized function $t_H(G) = h_H(G)/|G|^{|H|}$, which is the fraction of mappings $f : V(H) \rightarrow V(G)$ which are homomorphisms. Sidorenko's conjecture states that for every bipartite graph H with m edges and every graph G , $t_H(G) \geq t_{K_2}(G)^m$. We will prove that this is the case for H as in Theorem 1.

We use a probabilistic technique known as dependent random choice. The idea is that most small subsets of the neighborhood of a random vertex have large common neighborhood. Our first lemma gives a counting version of this technique. We will then combine this with a simple embedding lemma to give a lower bound for $t_H(G)$ in terms of $t_{K_2}(G)$. For a vertex v in a graph G , the *neighborhood* $N(v)$ is the set of vertices adjacent to v . For a sequence S of vertices of a graph G , the *common neighborhood* $N(S)$ is the set of vertices adjacent to every vertex in S .

Lemma 1 *Let G be a graph with N vertices and $pN^2/2$ edges. Call a vertex v bad with respect to k if the number of sequences of k vertices in $N(v)$ with at most $(2n)^{-n-1}p^kN$ common neighbors is at least $\frac{1}{2n}|N(v)|^k$. Call v good if it is not bad with respect to k for all $1 \leq k \leq n$. Then the sum of the degrees of the good vertices is at least $pN^2/2$.*

Proof: We write $v \sim k$ to denote that v is bad with respect to k . Let X_k denote the number of pairs (v, S) with S a sequence of k vertices, v a vertex adjacent to every vertex in S , and $|N(S)| \leq (2n)^{-n-1}p^kN$. We have

$$(2n)^{-n-1}p^kN \cdot N^k \geq X_k \geq \sum_{v, v \sim k} \frac{1}{2n}|N(v)|^k \geq \frac{1}{2n}N \left(\sum_{v, v \sim k} |N(v)|/N \right)^k = \frac{1}{2n}N^{1-k} \left(\sum_{v, v \sim k} |N(v)| \right)^k.$$

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The first inequality is by summing over S , the second inequality is by summing over vertices v which are bad with respect to k , and the third inequality is by convexity of the function $f(x) = x^k$. We therefore get

$$\sum_{v, v \sim k} |N(v)| \leq (2n)^{-n/k} pN^2 \leq \frac{1}{2n} pN^2.$$

Hence,

$$\sum_{v, v \text{ good}} |N(v)| \geq \sum_v |N(v)| - \sum_{k=1}^n \sum_{v, v \sim k} |N(v)| \geq pN^2 - n \cdot \frac{1}{2n} pN^2 = pN^2/2,$$

as required. \square

Lemma 2 *Suppose \mathcal{H} is a hypergraph with v vertices and at most e edges and \mathcal{G} is a hypergraph on N vertices with the property that for each k , $1 \leq k \leq v$, the number of sequences of k vertices of \mathcal{G} that do not form an edge of \mathcal{G} is at most $\frac{1}{2e} N^k$. Then the number of homomorphisms from \mathcal{H} to \mathcal{G} is at least $\frac{1}{2} N^v$.*

Proof: Consider a random mapping of the vertices of \mathcal{H} to the vertices of \mathcal{G} . The probability that a given edge of \mathcal{H} does not map to an edge of \mathcal{G} is at most $\frac{1}{2e}$. By the union bound, the probability that there is an edge of \mathcal{H} that does not map to an edge of \mathcal{G} is at most $e \cdot \frac{1}{2e} = 1/2$. Hence, with probability at least $1/2$, a random mapping gives a homomorphism, so there are at least $\frac{1}{2} N^v$ homomorphisms from \mathcal{H} to \mathcal{G} . \square

Lemma 3 *Let $H = (V_1, V_2, E)$ be a bipartite graph with n vertices and m edges such that there is a vertex $u \in V_1$ which is adjacent to all vertices in V_2 . Let G be a graph with N vertices and $pN^2/2$ edges, so $t_{K_2}(G) = p$. Then the number of homomorphisms from H to G is at least $(2n)^{-n^2} p^m N^n$.*

Proof: Let $n_i = |V_i|$ for $i \in \{1, 2\}$. We will give a lower bound on the number of homomorphisms $f : V(H) \rightarrow V(G)$ that map u to a good vertex v of G . Suppose we have already picked $f(u) = v$. Let \mathcal{H} be the hypergraph with vertex set V_2 , where $S \subset V_2$ is an edge of \mathcal{H} if there is a vertex $w \in V_1 \setminus \{u\}$ such that $N(w) = S$. The number of vertices of \mathcal{H} is n_2 , which is at most n , and the number of edges of \mathcal{H} is $n_1 - 1$, which is also at most n . Let \mathcal{G} be the hypergraph on $N(v)$, where a sequence R of k vertices of $N(v)$ is an edge of \mathcal{G} if $|N(R)| \geq (2n)^{-n-1} p^k N$. Since v is good, for each k , $1 \leq k \leq v$, the number of sequences of k vertices of \mathcal{G} that are not the vertices of an edge of \mathcal{G} is at most $\frac{1}{2n} N^k$. Hence, by Lemma 2, there are at least $\frac{1}{2} |N(v)|^{n_2}$ homomorphisms g from \mathcal{H} to \mathcal{G} . Pick one such homomorphism g , and let $f(x) = g(x)$ for $x \in V_2$. By construction, once we have picked $f(u)$ and $f(V_2)$, there are at least $(2n)^{-n-1} p^{|N(w)|} N$ possible choices for $f(w)$ for each vertex $w \in V_1$. Hence, the number of homomorphisms from H to G is at least

$$\begin{aligned} \sum_{v \text{ good}} \frac{1}{2} |N(v)|^{n_2} \prod_{w \in V_1 \setminus \{u\}} (2n)^{-n-1} p^{|N(w)|} N &= \frac{1}{2} (2n)^{-(n-1)(n_1-1)} p^{m-n_2} N^{n_1-1} \sum_{v \text{ good}} |N(v)|^{n_2} \\ &\geq \frac{1}{2} (2n)^{-(n-1)(n_1-1)} p^{m-n_2} N^{n_1-1} N \left(\sum_{v \text{ good}} |N(v)|/N \right)^{n_2} \\ &\geq \frac{1}{2} (2n)^{-(n-1)(n_1-1)} p^{m-n_2} N^{n_1} (pN/2)^{n_2} \\ &\geq (2n)^{-n^2} p^m N^n. \end{aligned}$$

The first inequality is by convexity of the function $q(x) = x^k$ and the second inequality is by the lower bound on the sum of the degrees of good vertices given by Lemma 1. \square

We next complete the proof of Theorem 1 by improving the bound in the previous lemma on the number of homomorphisms from H to G using a tensor power trick. The tensor product $F \times G$ of two graphs F and G has vertex set $V(F) \times V(G)$ and any two vertices (u_1, u_2) and (v_1, v_2) are adjacent in $F \times G$ if and only if u_i is adjacent with v_i for $i \in \{1, 2\}$. Let $G^1 = G$ and $G^r = G^{r-1} \times G$. Note that $t_H(F \times G) = t_H(F) \times t_H(G)$ for all H, F, G .

Proof of Theorem 1: Suppose for contradiction that there is a graph G such that $t_H(G) < t_{K_2}(G)^m$. Denote the number of edges of G as $pN^2/2$, so $t_{K_2}(G) = p$. Let $c = \frac{t_H(G)}{t_{K_2}(G)^m} < 1$. Let r be such that $c^r < (2n)^{-n^2}$. Then

$$t_H(G^r) = t_H(G)^r = c^r t_{K_2}(G)^{mr} = c^r t_{K_2}(G^r)^m < (2n)^{-n^2} t_{K_2}(G^r)^m.$$

However, this contradicts Lemma 3 applied to H and G^r . This completes the proof. \square

References

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