Ramsey numbers of sparse hypergraphs

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Abstract

We give a short proof that any k-uniform hypergraph H on n vertices with bounded degree Δ has Ramsey number at most $c(\Delta, k)n$, for an appropriate constant $c(\Delta, k)$. This result was recently proved by several authors, but those proofs are all based on applications of the hypergraph regularity method. Here we give a much simpler, self-contained proof which uses new techniques developed recently by the authors together with an argument of Kostochka and Rödl. Moreover, our method demonstrates that, for $k \geq 4$,

$$c(\Delta, k) \leq 2^{2^{\cdot^{\cdot^{\cdot^{2^{c\Delta}}}}}},$$

where the tower is of height k and the constant c depends on k. It significantly improves on the Ackermann-type upper bound that arises from the regularity proofs, and we present a construction which shows that, at least in certain cases, this bound is not far from best possible. Our methods also allows us to prove quite sharp results on the Ramsey number of hypergraphs with at most m edges.

1   Introduction

For a graph H, the *Ramsey number* $r(H)$ is the least positive integer N such that, in every two-colouring of the edges of complete graph K_N on N vertices, there is a monochromatic copy of H. Ramsey's theorem states that $r(H)$ exists for every graph H. A classical result of Erdős and Szekeres, which is a quantitative version of Ramsey's theorem, implies that $r(K_k) \leq 2^{2k}$ for every positive integer k. Erdős showed using probabilistic arguments that $r(K_k) > 2^{k/2}$ for $k > 2$. Over the last sixty years, there have been several improvements on these bounds (see, e.g., [4]). However, despite efforts by various researchers, the constant factors in the above exponents remain the same.

Determining or estimating Ramsey numbers is one of the central problems in combinatorics, see the book *Ramsey theory* [17] for details. Besides the complete graph, the next most classical topic in this area concerns the Ramsey numbers of sparse graphs, i.e., graphs with certain upper bound constraints on the degrees of the vertices. The study of these Ramsey numbers was initiated by Burr and Erdős in 1975, and this topic has since played a central role in graph Ramsey theory. Burr and Erdős conjectured, and it was proved by Chvátal, Rödl, Szemerédi and Trotter [3], that for every graph G on n vertices and maximum degree Δ,

$$r(G) \leq c(\Delta)n.$$

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Their proof of this theorem is a classic application of Szemerédi's beautiful regularity lemma. However, the use of this lemma makes the upper bound on $c(\Delta)$ grow as a tower of 2s with height exponential in Δ. Eaton [8] used a variant of the regularity lemma to obtain the upper bound $c(\Delta) \leq 2^{2^{c\Delta}}$ for some fixed c. A novel approach of Graham, Rödl, Rucinski [15] that did not use any form of the regularity lemma gives the upper bound $c(\Delta) \leq 2^{c\Delta \log^2 \Delta}$ for some fixed c. In the other direction, in [16] they proved that there is a positive constant c such that, for every $\Delta \geq 2$ and $n \geq \Delta + 1$, there is a bipartite graph G with n vertices and maximum degree at most Δ satisfying $r(G) \geq 2^{c\Delta}n$. Recently, the authors [5], [12] closed the gap for bipartite graphs by showing that, for every bipartite graph G with n vertices and maximum degree Δ, $r(G) \leq 2^{c\Delta}n$ for some fixed c.

A *hypergraph* $H = (V, E)$ consists of a vertex set V and an edge set E, which is a collection of subsets of V. A hypergraph is *k-uniform* if each edge has exactly k vertices. The *Ramsey number $r(H)$* of a k-uniform hypergraph H is the smallest number N such that, in any 2-colouring of the edges of the complete k-uniform hypergraph $K_N^{(k)}$, there is guaranteed to be a monochromatic copy of H. The existence of these numbers was proven by Ramsey [23], but no proper consideration of the values of these numbers was made until the paper by Erdős and Rado [11]. To understand the growth of Ramsey numbers for hypergraphs, it is useful to introduce the tower function $t_i(x)$, which is defined by $t_1(x) = x$ and $t_{i+1}(x) = 2^{t_i(x)}$, i.e.,

$$t_{i+1}(x) = 2^{2^{\cdot^{\cdot^{\cdot^{2^x}}}}},$$

where the number of 2s in the tower is i. Erdős and Rado showed that for H being the complete k-uniform hypergraph $K_l^{(k)}$, $r(H) \leq t_k(cl)$, where the constant c depends on k. In the other direction, Erdős and Hajnal (see [17]) proved that for $H = K_l^{(k)}$, $r(H) \geq t_{k-1}(cl^2)$, where the constant c depends on k.

One can naturally try to extend the sparse graph Ramsey results to hypergraphs. Kostochka and Rödl [19] showed that for every $\epsilon > 0$, the Ramsey number of any k-uniform hypergraph H with n vertices and maximum degree Δ satisfies

$$r(H) \leq c(\Delta, k, \epsilon)n^{1+\epsilon},$$

where $c(\Delta, k, \epsilon)$ only depends on Δ, k, and ϵ. Since the first proof of the sparse graph Ramsey theorem used Szemerédi's regularity lemma, it was therefore natural to expect that, given the recent advances in developing a hypergraph regularity method [14, 24, 22], linear bounds might as well be provable for hypergraphs. Such a program was indeed recently pursued by several authors (Cooley, Fountoulakis, Kühn, and Osthus [6, 7]; Nagle, Olsen, Rödl, and Schacht [21]), with the result that we now have the following theorem:

Theorem 1 *Let Δ and k be positive integers. Then there exists a constant $c(\Delta, k)$ such that the Ramsey number of any k-uniform hypergraph H with n vertices and maximum degree Δ satisfies*

$$r(H) \leq c(\Delta, k)n.$$

In this paper we will give a short proof of this theorem, which is much simpler and avoids all use of the regularity lemma, building instead on techniques developed recently by Conlon [5] and by Fox and Sudakov [12] in order to study embeddings of sparse bipartite graphs in dense graphs.

The first main result of this paper is an extension of this work from graphs to hypergraphs. An l-uniform hypergraph is *l-partite* if there is a partition of the vertex set into l parts such that each edge

has exactly one vertex in each part. We prove the following Turán-type result for l-uniform l-partite hypergraphs:

Theorem 2 *There exists a constant $c = c(l)$ such that if F is an l-uniform l-partite hypergraph with n vertices and maximum degree Δ and G is an l-uniform l-partite hypergraph with parts of size $N \geq (\epsilon/2)^{-c\Delta^{l-1}} n$ and at least ϵN^l edges, then G contains a copy of F.*

Then, in Section 3, we will prove Theorem 1 by applying an argument of Kostochka and Rödl which shows that the Ramsey problem for general hypergraphs may be reduced to an application of the Turán theorem in the l-uniform l-partite case. This argument combined with our Theorem 2 shows that, for $k \geq 4$ and k-uniform hypergraph H with n vertices and maximum degree Δ,

$$r(H) \leq t_k(c\Delta)n,$$

where the constant c depends on k. For $k = 3$, the proof shows that $r(H) \leq t_3(c\Delta \log \Delta)n$. This is clearly much better than the Ackermann-type upper bound that arises from the regularity proofs. The tower-type upper bound cannot be avoided as demonstrated by the lower bound of Erdős and Hajnal for the Ramsey number of the complete k-uniform hypergraph on n vertices. This hypergraph has maximum degree $\Delta = \binom{n-1}{k-1}$ and Ramsey number at least $t_{k-1}(c\Delta^{\frac{2}{k-1}})n$, where the constant c depends on k.

For k-uniform hypergraphs $H_1, \ldots, H_q$, the *multicolour Ramsey number* $r(H_1, \ldots, H_q)$ is the minimum N such that, in any q-colouring of the edges of the complete k-uniform hypergraph $K_N^{(k)}$ with colours $1, \ldots, q$, there is a monochromatic copy of H_i in colour i for some i, $1 \leq i \leq q$. The proof of Theorem 1 presented here extends in a straightforward manner to the multicolour generalisation, which states that for all positive integers Δ, k, and q, there exists a constant $c(\Delta, k, q)$ such that, if $H_1, \ldots, H_q$ are k-uniform hypergraphs each with n vertices and maximum degree Δ, then $r(H_1, \ldots, H_q) \leq c(\Delta, k, q)n$. The proof demonstrates that we may take $c(\Delta, k, q) \leq t_k(c\Delta)$ for $k \geq 4$ and $c(\Delta, 3, q) \leq t_3(c\Delta \log \Delta)$, where the constant c depends on k and q. In the other direction, in Section 4 we construct, for each sufficiently large Δ, a 3-uniform hypergraph H with maximum degree at most Δ for which the 4-colour Ramsey number of H satisfies $r(H, H, H, H) \geq t_3(c\Delta)n$, where n is the number of vertices of H. This example shows that our upper bound for hypergraph Ramsey numbers is probably close to being best possible.

The same example also shows that there is a 3-uniform hypergraph H with m edges for which the 4-colour Ramsey number of H is at least $t_3(c\sqrt{m})$. On the other hand, one can easily deduced from the proof of Theorem 1 that for any k-uniform hypegraph H with m edges, we have that the q-colour Ramsey number of H satisfies $r(H, \cdots, H) \leq t_3(c\sqrt{m} \log m)$ for $k = 3$, and $r(H, \cdots, H) \leq t_k(c\sqrt{m})$ for $k \geq 4$, where c depends on k and q.

2   A Turán theorem for l-uniform l-partite hypergraphs

The following is a generalisation to hypergraphs of a lemma which has appeared increasingly in the literature on Ramsey theory, whose proof uses a probabilistic argument known as *dependent random choice*. Early versions of this technique were developed in the papers [13], [18], [25]. Later, variants were discovered and applied to various Ramsey and density-type problems (see, e.g., [20, 1, 26, 19, 12, 5]). We define the *weight $w(S)$* of a set S of edges in a hypergraph to be the size of the union of these edges, i.e., the number of vertices that are contained in at least one of these edges.

Lemma 1 *Suppose s, Δ are positive integers, $\epsilon, \beta > 0$, and $G_r = (V_1, \cdots, V_r; E)$ is an r-uniform r-partite hypergraph with $|V_1| = |V_2| = \cdots = |V_r| = N$ and at least ϵN^r edges. Then there exists an $(r-1)$-uniform $(r-1)$-partite hypergraph G_{r-1} on the vertex sets $V_2, \cdots, V_r$ which has at least $\frac{\epsilon^s}{2} N^{r-1}$ edges and such that for each nonnegative integer $w \leq (r-1)\Delta$, there are at most $4r\Delta\epsilon^{-s}\beta^s w^{r\Delta} r^w N^w$ dangerous sets of edges of G_{r-1} with weight w, where a set S of edges of G_{r-1} is dangerous if $|S| \leq \Delta$ and the number of vertices $v \in V_1$ such that for every edge $e \in S$, $e + v \in G_r$ is less than βN.*

Proof: Let $\mathcal{C}$ be the complete $(r-1)$-uniform $(r-1)$-partite hypergraph on the vertex sets $V_2, \cdots, V_r$. For any edge e in $\mathcal{C}$, let $d(e)$ be the degree of e in G_r, i.e., the number of vertices in V_1 such that $e + v \in G_r$. Let T be a set of s random vertices of V_1, chosen uniformly with repetitions. Let A be the set of edges in $\mathcal{C}$ which are common neighbours of the vertices of T, i.e., an edge e of $\mathcal{C}$ is in A if $e + v$ is an edge of G_r for all $v \in T$. Let X denote the cardinality of A. We will show that with positive probability, the set A will be the edge set of a hypergraph G_{r-1} on vertex sets $V_2, \ldots, V_r$ with the desired properties. By linearity of expectation and by convexity of $f(z) = z^s$,

$$
\begin{aligned}
\mathbb{E}[X] &= \sum_{e \in \mathcal{C}} \mathbb{P}[e \in A] = \sum_{e \in \mathcal{C}} \left(\frac{d(e)}{N} \right)^s \\
&\geq \frac{N^{r-1} \left(\frac{\sum_{e \in \mathcal{C}} d(e)}{N^{r-1}} \right)^s}{N^s} \geq \frac{N^{r-1}(\epsilon N)^s}{N^s} = \epsilon^s N^{r-1}.
\end{aligned}
$$

Note that $X \leq N^{r-1}$ since $\mathcal{C}$ has N^{r-1} edges. Letting p denote the probability that $X \geq \mathbb{E}[X]/2$, we have

$$
\mathbb{E}[X] \leq (1-p)\mathbb{E}[X]/2 + pN^{r-1} \leq \mathbb{E}[X]/2 + pN^{r-1}.
$$

So the probability p that $X \geq \mathbb{E}[X]/2 \geq \epsilon^s N^{r-1}/2$ satisfies $p \geq \frac{\mathbb{E}[X]}{2N^{r-1}} \geq \epsilon^s/2$.

The number of subsets S of $V_2 \cup \ldots \cup V_r$ of size w is $\binom{(r-1)N}{w}$. For a given w-set S, the number of collections $\{e_1, \ldots, e_t\}$ of size t with $|e_i| = r-1$, and $e_i \subset S$ for $1 \leq i \leq t$ is $\binom{\binom{w}{r-1}}{t}$. Hence, summing over all nonnegative $t \leq \Delta$, the number of sets of edges of $\mathcal{C}$ with weight w and size at most Δ is at most

$$
\sum_{t=0}^{\Delta} \binom{\binom{w}{r-1}}{t} \binom{(r-1)N}{w} \leq w^{r\Delta}(rN)^w = w^{r\Delta} r^w N^w.
$$

Let Y_w denote the random variable counting the number of dangerous sets S of edges of G_{r-1} with weight w. We next give an upper bound on $\mathbb{E}[Y_w]$. For a given set S of edges of $\mathcal{C}$, the probability that S is a subset of edges of G_{r-1} is $\left(\frac{|N(S)|}{N} \right)^s$, where $N(S)$ denotes the set of vertices $v \in V_1$ with $v + e$ an edge of G_r for all $e \in S$. So if S satisfies $|N(S)| < \beta N$, then the probability that S is a subset of edges of G_{r-1} is less than β^s. By linearity of expectation, we have $\mathbb{E}[Y_w] < \beta^s w^{r\Delta} r^w N^w$.

Let $\alpha = 4r\Delta\epsilon^{-s}$. Since Y_w is a nonnegative random variable, Markov's inequality implies that $\mathbb{P}(Y_w \geq \alpha\mathbb{E}[Y_w]) \leq \frac{1}{\alpha}$. Hence, the probability that there is a nonnegative integer $w \leq (r-1)\Delta$ with $Y_w \geq \alpha\beta^s w^{r\Delta} r^w N^w$ is at most $r\Delta/\alpha = \epsilon^s/4$. Since the probability that $X \geq \frac{\epsilon^s}{2} N^{r-1}$ is at least $\epsilon^s/2$, we can satisfy the conditions of the lemma with probability at least $\epsilon^s/4$. $\qquad\square$

By simply iterating the previous lemma $l-1$ times, we obtain the following corollary.

Corollary 1 *Suppose s, Δ are positive integers, $\epsilon, \beta > 0$, and $G_l = (V_1, \cdots, V_l; E_l)$ is an l-uniform l-partite hypergraph with $|V_1| = |V_2| = \cdots = |V_l| = N$ and at least ϵN^l edges. Let $\delta_l = \epsilon$ and*

$\delta_{r-1} = \delta_r^s/2$ for $2 \leq r \leq l$. Then, for $1 \leq r \leq l-1$, there are r-uniform r-partite hypergraphs $G_r = (V_{l-r+1}, \ldots, V_l, E_r)$ with the following properties:

1. G_r has at least $\delta_r N^r$ edges for $1 \leq r \leq l$, and

2. for $2 \leq r \leq l$ and each nonnegative integer $w \leq (r-1)\Delta$, there are at most $4r\Delta\delta_r^{-s}\beta^s w^{r\Delta}r^w N^w$ dangerous sets of G_{r-1} with weight w, where a set S of edges of G_{r-1} is dangerous if $|S| \leq \Delta$ and the number of vertices $v \in V_{l-r+1}$ such that for every edge $e \in S$, $e + v \in G_r$ is less than βN.

This is all the preparation we need before proving our main contribution, Theorem 2. For the proof, we will use Corollary 1 and then show how to embed F into G. The latter part is closely related to the many embedding results proven by Fox and Sudakov in [12]. We will actually prove the following more precise version of Theorem 2:

Theorem 3 *Let $l \geq 3$, F be an l-uniform l-partite hypergraph, on vertex sets $W_1, \cdots, W_l$, with at most n vertices and maximum degree Δ. Let G_l be an l-uniform l-partite graph, on vertex sets $V_1, \cdots, V_l$ with $|V_1| = \cdots = |V_l| = N$, with at least ϵN^l edges. Then, provided that $N \geq (\epsilon/2)^{-(2l\Delta)^{l-1}} n$, G_l contains a copy of F.*

Proof: We apply Corollary 1 with $s = 2l\Delta$, $\delta_l = \epsilon$, $\delta_{i-1} = \delta_i^s/2$ for $2 \leq i \leq l$, and $\beta = 2(\epsilon/2)^{(2l\Delta)^{l-1}}$ to get hypergraphs $G_{l-1}, \ldots, G_1$ with the properties guaranteed by Corollary 1. It is easy to check by induction on i that $\delta_{l-i} = 2^{-(s^i-1)/(s-1)}\epsilon^{s^i}$, so

$$\delta_1 = 2^{-(s^{l-1}-1)/(s-1)}\epsilon^{s^{l-1}} \geq 2(\epsilon/2)^{(2l\Delta)^{l-1}} = \beta$$

and $\delta_1 N \geq \beta N \geq 2n$.

We now construct an $l\Delta$-uniform *bad hypergraph* B with vertex set $V_1 \cup \ldots \cup V_l$ where each edge of B has exactly Δ vertices in each V_i. A set $T \subset V_1 \cup \ldots \cup V_l$ which contains exactly Δ vertices in each V_i is an edge of B if and only if there is a dangerous set S of edges of G_r for some r, $1 \leq r \leq l-1$, such that the union of the edges of S is a subset of T. In other words, an edge of B is just an extension of the union of the edges of a dangerous set. For a particular dangerous set S of edges with weight w in some G_r, the number of edges of B that are extensions of the union of the edges in S is at most $N^{l\Delta-w}$ since there are at most N ways to pick each of the $l\Delta - w$ remaining vertices that make up an edge. Summing over all r and w, and using the fact that $l \geq 3$ and, for $r \geq 2$, $\delta_r^s \geq \delta_2^s = 2\delta_1 \geq 2\beta$, the number of edges of B is at most

$$
\begin{aligned}
\sum_{r=2}^{l} \sum_{w=0}^{(l-1)\Delta} 4r\Delta\delta_r^{-s}\beta^s w^{r\Delta}r^w N^{l\Delta} &\leq l^2\Delta \cdot 2l\Delta\beta^{s-1}(l\Delta)^{l\Delta}l^{l\Delta}N^{l\Delta} = 2l^3\Delta^2(l^2\Delta)^{l\Delta}\beta^{l\Delta-1}\beta^{l\Delta}N^{l\Delta}\\
&\leq 2^{1+3(l\Delta)^2}\beta^{l\Delta-1}\beta^{l\Delta}N^{l\Delta} \leq 2^{1+3(l\Delta)^2}2^{(1-(2l\Delta)^{l-1})(l\Delta-1)}\beta^{l\Delta}N^{l\Delta}\\
&\leq 2^{-4(l\Delta)^2}\beta^{l\Delta}N^{l\Delta} < \left(\frac{\beta}{4l\Delta}\right)^{l\Delta}\binom{N}{l\Delta}.
\end{aligned}
$$

Call a set $U \subset V_1 \cup \ldots \cup V_l$ with at most Δ vertices in each V_i *bad* if there are at least

$$\left(\frac{\beta}{4l\Delta}\right)^{l\Delta-|U|}\binom{N}{l\Delta-|U|}$$

5

edges of B that contain U; otherwise call U *good*. Note that the above calculation on the number of edges of B demonstrates that the empty set is good. We next prove the following important claim.

Claim 1 *If S is a dangerous set of edges in G_r for some r, $1 \le r \le l - 1$, and U is a good set, then the union of the edges in S is not a subset of U.*

Proof: Suppose for contradiction that the union of the edges in S is a subset of U. The number of extensions of U to a set which contains exactly Δ vertices in each V_i is

$$\prod_{i=1}^{l} \binom{N - |V_i \cap U|}{\Delta - |V_i \cap U|}$$

since we can pick for each i any $\Delta - |V_i \cap U|$ vertices of $V_i \setminus U$ to extend U. By definition, all of these sets are edges in B. Using the simple fact that if $x_1, \ldots, x_l$ are nonnegative integers then $\prod_{i=1}^{l} x_i! \le (\sum_{i=1}^{l} x_i)!$, it is straightforward to check that

$$
\begin{aligned}
\prod_{i=1}^{l} \binom{N - |V_i \cap U|}{\Delta - |V_i \cap U|} &\ge (N/2)^{l\Delta - |U|} \prod_{i=1}^{l} (\Delta - |V_i \cap U|)!^{-1} \ge \left(\frac{1}{2}\right)^{l\Delta - |U|} \binom{N}{l\Delta - |U|} \\
&\ge \left(\frac{\beta}{4l\Delta}\right)^{l\Delta - |U|} \binom{N}{l\Delta - |U|},
\end{aligned}
$$

which contradicts U being good. $\square$

Given a good set U with $|V_i \cap U| < \Delta$ and $v \in V_i \setminus U$, we say v is *bad with respect to U* if $U \cup \{v\}$ is bad. Let B_U denote the set of vertices that are bad with respect to U. We will show that for U good we have $|B_U| \le \frac{\beta N}{4l\Delta}$. Indeed, suppose $|B_U| > \frac{\beta N}{4l\Delta}$. Then the number of pairs (T, v) with T an edge of B, $U \subset T$, $v \in T \setminus U$, and $U \cup \{v\}$ is bad is at least

$$|B_U| \left(\frac{\beta}{4l\Delta}\right)^{l\Delta - |U| - 1} \binom{N}{l\Delta - |U| - 1}.$$

Therefore, the number of edges of B containing U is at least

$$\frac{|B_U|}{l\Delta - |U|} \left(\frac{\beta}{4l\Delta}\right)^{l\Delta - |U| - 1} \binom{N}{l\Delta - |U| - 1} > \left(\frac{\beta}{4l\Delta}\right)^{l\Delta - |U|} \binom{N}{l\Delta - |U|},$$

contradicting the fact that U is good.

Fix a labeling $\{v_1, \cdots, v_n\}$ of the vertices of F such that all vertices in W_{i+1} precede all those in W_i for all $i = 1, \cdots, l - 1$. For each i, let $L_i = \{v_1, \cdots, v_i\}$. For each vertex v_h, the *trace neighbourhood* $N(v_h)$ is the set of vertices v_m with $m < h$ that are in an edge of F with v_m. Note that $N(v_h)$ contains at most Δ vertices in each W_r since F has maximum degree Δ. We will find an embedding f of the vertices of F such that $f(W_r) \subset V_r$ for $1 \le r \le l$ and for each $i \le n$,

1. $f(N(v) \cap L_i)$ is good for each vertex v of F, and

2. $f(e \cap L_i)$ is an edge of G_r for each edge e of F, where $r = |e \cap L_i|$.

The proof will be complete once we find such an embedding f since, for each edge e of F, $f(e \cap L_n) = f(e)$ is an edge of G_l, so f provides an embedding of F in G_l. The embedding will be constructed one vertex at a time, in increasing order of subscript, so the proof will be by induction on i. As noted earlier, the empty set is good, so our base case $i = 0$ is satisfied.

Suppose then that at step i, we have found an embedding f of $v_1, \ldots, v_i$ such that

1. for each vertex v of F, $f(N(v) \cap L_i)$ is good, and

2. for each edge e of F, $f(e \cap L_i)$ is an edge of G_r, where $r = |e \cap L_i|$.

Let j be such that $v_{i+1} \in W_j$. Let $e_1, \ldots, e_d$ denote the edges of F that contain v_{i+1} and $e'_1, \ldots, e'_d$ denote the truncations of $e_1, \ldots, e_d$ by deleting all j vertices from each e_t that are in some W_h with $h \leq j$. Each e'_t consists of one vertex from each W_h with $h > j$. Also, $d \leq \Delta$ since F has maximum degree Δ.

Since F has maximum degree Δ, there are less than $l\Delta$ vertices v for which $v_{i+1} \in N(v)$. For each such v, $f(N(v) \cap L_i)$ is good, so there are at most $\frac{\beta}{4l\Delta} N$ vertices w in V_j for which $f(N(v) \cap L_i) \cup w$ is bad. Adding over all such v, we conclude that there are at most $\frac{\beta}{4} N$ bad vertices in all associated with v_{i+1}.

Suppose we are still embedding vertices of W_l in V_l. Since the edge set of G_1 is just a subset of V_l whose size by Corollary 1 is at least $\delta_1 N = \beta N$, then we can choose any of these at least βN vertices other than $f(v_1), \ldots, f(v_i)$ for $f(v_{i+1})$ to satisfy the second of the two desired properties for $f(v_{i+1})$. We see that there are at least $\beta N - i - \frac{\beta}{4} N > \frac{3\beta N}{4} - n > 0$ vertices to choose from for $f(v_{i+1})$ to satisfy both of the desired properties.

If, now, we have chosen all of the vertices in $W_l, \cdots, W_{j+1}$ and we are trying to embed vertex v_{i+1} in W_j (we may have already embedded other vertices in W_j), we can do so. To see this, by the induction hypothesis, $f(N(v_{i+1}) \cap L_i) = f(N(v_{i+1})) = \bigcup_{t=1}^d f(e'_t)$ is good. By Claim 1, this implies that the set $\{f(e'_1), \ldots, f(e'_d)\}$ of edges of G_{l-j} is not dangerous, i.e., there are at least βN vertices $v \in V_j$ such that $f(e'_t) \cup v$ is an edge of G_{l-j+1} for $1 \leq t \leq d$. Therefore, since there are at most $\frac{\beta}{4} N$ bad vertices associated with v_{i+1} and we have already chosen $f(v_1), \ldots, f(v_i)$, we have at least $\frac{3}{4}\beta N - i > \frac{3}{4}\beta N - n > 0$ choices for $f(v_{i+1})$, which completes the proof. $\qquad \square$

3 The Ramsey theorem

We are now ready to prove Theorem 1 in the following form:

Theorem 4 *Let Δ and $k \geq 3$ be positive integers. Then the Ramsey number of any k-uniform hypergraph H with n vertices and maximum degree Δ satisfies*

$$r(H) \leq r_k(k\Delta)^{(2k\Delta^2)^{k\Delta}} n,$$

where $r_k(l) = r(K_l^{(k)})$.

Proof: We use the argument of Kostochka and Rödl [19] together with Theorem 3. Let $l = (k - 1)\Delta + 1$. Suppose we have a red-blue colouring of the complete k-uniform hypergraph on N vertices.

Let G be the hypergraph consisting of all the red edges and let $r_k(l)$ be the Ramsey number of the hypergraph $K_l^{(k)}$. Then, in each subset of the vertices of size $r_k(l)$, there is at least one monochromatic $K_l^{(k)}$. Counting over all such sets and dividing out by possible multiple counts we see that we have at least

$$\frac{\binom{N}{r_k(l)}}{\binom{N-l}{r_k(l)-l}} \geq \frac{N^l}{r_k(l)^l}$$

monochromatic $K_l^{(k)}$. Therefore, either G or its complement $\overline{G}$ contains at least $N^l/2r_k(l)^l$ cliques $K_l^{(k)}$. We will suppose that it is G.

Consider now the l-uniform hypergraph $G^{(l)}$ on the same vertex set as G whose edges are exactly those l-tuples which form a complete $K_l^{(k)}$ in G. This hypergraph has at least $N^l/2r_k(l)^l$ edges. Partition its vertex set randomly into l parts $V_1, \cdots, V_l$ of equal size N/l. The total number of partitions is $\frac{N!}{(N/l)!^l}$ and, for any given edge e, there are $l!\frac{(N-l)!}{(N/l-1)!^l}$ partitions such that each vertex of this edge is in a different part of the partition. Therefore, the expected number of edges with one vertex in each set of the random partition is at least

$$e(G^{(l)}) \left(l!\frac{(N-l)!}{(N/l-1)!^l} \Big/ \frac{N!}{(N/l)!^l} \right) \geq \frac{N^l}{2r_k(l)^l} \cdot l!\frac{(N-l)!}{(N/l-1)!^l}\frac{(N/l)!^l}{N!} \geq \frac{N^l}{2r_k(l)^l}\frac{l!}{l^l} = \frac{l!}{2r_k(l)^l}\left(\frac{N}{l}\right)^l.$$

Now choose such a partition and let $\hat{G}^{(l)}$ be the l-uniform l-partite subhypergraph of $G^{(l)}$ consisting of those edges of $G^{(l)}$ which have one edge in each partite set. Note that $\hat{G}^{(l)}$ has N/l vertices in each part and at least $\epsilon \left(\frac{N}{l}\right)^l$ edges, where $\epsilon = \frac{l!}{2r_k(l)^l}$.

Now we extend H to an l-uniform l-partite hypergraph $H^{(l)}$. We first note that the vertices of H can be partitioned into l subsets $A_1, \ldots, A_l$ such that each edge of H has at most one vertex in each part. This is equivalent to saying that the graph H' with the same vertex set as H and with two vertices adjacent if they lie in an edge of H has chromatic number at most l. Since H' has maximum degree at most $(k-1)\Delta$, it has chromatic number at most $(k-1)\Delta + 1 = l$. For each edge e of H, we add one auxiliary vertex to each A_i which is disjoint from e (in total $l - k$ vertices). Note that the maximum degree of $H^{(l)}$ remains Δ. The total number of auxiliary vertices added is at most $\frac{\Delta n}{k} \cdot (l - k) < \Delta(\Delta - 1)n$ since there are $l - k$ auxiliary vertices for each edge and the total number of edges of H is at most $\frac{\Delta n}{k}$. Hence, $H^{(l)}$ has less than $\Delta^2 n$ vertices.

Applying Theorem 3 with $F = H^{(l)}$, $G_l = \hat{G}^{(l)}$, and $\epsilon = \frac{l!}{2r_k(l)^l}$ we see that, provided

$$\frac{N}{l} \geq (\epsilon/2)^{-(2l\Delta)^{l-1}} \cdot \Delta^2 n,$$

$\hat{G}^{(l)}$ contains a copy of $H^{(l)}$. But now, by the construction of $H^{(l)}$, this implies that every edge in H is contained inside an edge of $\hat{G}^{(l)}$. But $\hat{G}^{(l)}$ was chosen in such a way that every k-tuple within any edge of $\hat{G}^{(l)}$ is an edge in G. Therefore G contains a copy of H, so we are done. $\qquad \square$

As mentioned in the introduction, the proof of Theorem 1 presented here extends in a straightforward manner to the following multicolour generalisation.

Theorem 5 *For all positive integers Δ, k, and q, there exists a constant $c(\Delta, k, q)$ such that, if $H_1, \ldots, H_q$ are k-uniform hypergraphs each with n vertices and maximum degree Δ, then*

$$r(H_1, \ldots, H_q) \leq c(\Delta, k, q)n.$$

The only difference in the proof is in Theorem 4, where we replace $r_k(l)$ by $r_k(l;q)$, the q-colour Ramsey number for the complete k-uniform hypergraph on l vertices. Erdős and Rado [11] showed that $r_k(l;q) \leq t_k(cl)$, where the constant c depends on k and q. We therefore may take $c(\Delta, k, q) \leq t_k(c\Delta)$ for $k \geq 4$ and $c(\Delta, 3, q) \leq t_3(c\Delta \log \Delta)$, where the constant c depends on k and q.

Remark: The *strong chromatic number* of a hypergraph H is the minimum number of colors required to colour the vertices of H so that each edge of H has no repeated colour. The proof of Theorem 4 demonstrates that if H is a k-uniform hypergraph with n vertices, maximum degree Δ, and strong chromatic number l, then the q-colour Ramsey number of H satisfies

$$r(H, \cdots, H) \leq r_k(l;q)^{(2l\Delta)^l}.$$

Indeed, in the proof of Theorem 4, we only used the fact that the vertices of H can be partitioned into l parts such that every edge has at most one vertex in each part.

4   Lower bound construction

The following theorem demonstrates that our upper bound for hypergraph Ramsey numbers proved in the previous section in some cases cannot be improved significantly.

Theorem 6 *There is $c > 0$ such that for each sufficiently large Δ, there is a 3-uniform hypergraph H with maximum degree at most Δ for which the 4-colour Ramsey number of H satisfies*

$$r(H, H, H, H) \geq 2^{2^{c\Delta}} n,$$

where n is the number of vertices of H.

Proof: Our proof uses the same 4-edge-colouring of the complete 3-uniform hypergraph that was constructed by Erdős and Hajnal (see, e.g., [17]). Not only does this colouring have no large monochromatic complete 3-uniform hypergraph, but we show it also does not have any monochromatic copies of a much sparser 3-uniform hypergraph H.

Let $n \geq 4$ be even, $m = \lceil 2^{n/4} \rceil$, and suppose the edges of the complete graph K_m are coloured red or blue in such a way that neither colour contains a monochromatic copy of the graph $K_{n/2}$. Such an edge-colouring exists by the lower bound of Erdős (see [17]) on the Ramsey number of the complete graph.

Let $V = \{v_1, \cdots, v_n\}$ be a set of vertices and let H be the 3-uniform hypergraph on V whose edge set is given by $\{v_i, v_{i+1}, v_j\}$ for all $1 \leq i, j \leq n$. (Note that when $i = n$, we consider $i + 1$ to be equal to 1.) It is straightforward to check that every vertex in H has degree $\Delta \leq 3n$.

We are going to define a 4-colouring of the complete 3-uniform hypergraph on the set

$$T = \{(\gamma_1, \cdots, \gamma_m) : \gamma_i = 0 \text{ or } 1\}$$

in such a way that there is no monochromatic copy of H. Note that then we will be done, since T has size $2^m \geq 2^{2^{n/4}}$ while H has maximum degree $\Delta \leq 3n$.

To define our colouring, we need some definitions:

If $\epsilon = (\gamma_1, \cdots, \gamma_m)$, $\epsilon' = (\gamma_1', \cdots, \gamma_m')$ and $\epsilon \neq \epsilon'$, define

$$\delta(\epsilon, \epsilon') = \max\{i : \gamma_i \neq \gamma_i'\},$$

that is, $\delta(\epsilon, \epsilon')$ is the largest coordinate at which they differ. We can now define an ordering on T by

$$\epsilon < \epsilon' \text{ if } \gamma_\delta = 0, \gamma_\delta' = 1,$$

$$\epsilon' < \epsilon \text{ if } \gamma_\delta = 1, \gamma_\delta' = 0,$$

where $\delta = \delta(\epsilon, \epsilon')$. Another way of looking at this ordering is to assign to each ϵ the number $b(\epsilon) = \sum_{i=1}^m \gamma_i 2^{i-1}$. The ordering then says simply that $\epsilon < \epsilon'$ iff $b(\epsilon) < b(\epsilon')$.

It is important to note the following two properties of the function δ:

(a) if $\epsilon_1 < \epsilon_2 < \epsilon_3$, then $\delta(\epsilon_1, \epsilon_2) \neq \delta(\epsilon_2, \epsilon_3)$;
(b) if $\epsilon_1 < \epsilon_2 < \cdots < \epsilon_r$, then $\delta(\epsilon_1, \epsilon_r) = \max_{1 \leq i \leq r-1} \delta(\epsilon_i, \epsilon_{i+1})$.

Now we are ready to define our colouring of the complete 3-uniform hypergraph τ on vertex set T. To begin, suppose that $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ with $\epsilon_1 < \epsilon_2 < \epsilon_3$ is an edge in τ. Write $\delta_1 = \delta(\epsilon_1, \epsilon_2), \delta_2 = \delta(\epsilon_2, \epsilon_3)$. Then we colour as follows:

$$C_1, \text{ if } \{\delta_1, \delta_2\} \text{ is red and } \delta_1 < \delta_2;$$
$$C_2, \text{ if } \{\delta_1, \delta_2\} \text{ is red and } \delta_1 > \delta_2;$$
$$C_3, \text{ if } \{\delta_1, \delta_2\} \text{ is blue and } \delta_1 < \delta_2;$$
$$C_4, \text{ if } \{\delta_1, \delta_2\} \text{ is blue and } \delta_1 > \delta_2,$$

where red and blue refer to the edge-colouring of K_m.

Now, let $S = \{\epsilon_1, \cdots, \epsilon_n\}_<$ be an ordered n-tuple (that is, $\epsilon_1 < \epsilon_2 < \ldots, < \epsilon_n$) within τ and suppose that there is a copy of H on S which is coloured by C_1. Suppose that the natural cycle $\{v_1, \cdots, v_n\}$ associated with H occurs as $\{\epsilon_{\pi(1)}, \cdots, \epsilon_{\pi(n)}\}$ where π is a permutation of $1, \cdots, n$. For each i, $1 \leq i \leq n$, let $\phi(i) = \max(\pi(i), \pi(i+1))$ and $\psi(i) = \min(\pi(i), \pi(i+1))$.

We claim that $\delta_{\phi(i)-1} = \delta(\epsilon_{\phi(i)-1}, \epsilon_{\phi(i)})$ must be larger than $\delta_j = \delta(\epsilon_j, \epsilon_{j+1})$ for all $j < \phi(i) - 1$. Suppose that $\psi(i) < \phi(i) - 1$ and consider the triple $\{\epsilon_{\psi(i)}, \epsilon_{\phi(i)-1}, \epsilon_{\phi(i)}\}_<$, which is an edge of the copy of H on S. The colouring C_1 implies that

$$\delta_{\phi(i)-1} = \delta(\epsilon_{\phi(i)-1}, \epsilon_{\phi(i)}) > \delta(\epsilon_{\psi(i)}, \epsilon_{\phi(i)-1}) = \max_{\psi(i) \leq j < \phi(i)-1} \delta_j.$$

This proves the claim for $\psi(i) \leq j < \phi(i) - 1$. Next consider the triple $\{\epsilon_j, \epsilon_{\psi(i)}, \epsilon_{\phi(i)}\}_<$ with $j < \psi(i)$, which is also an edge of the copy of H on S. The colouring C_1 implies that

$$\delta_j \leq \delta(\epsilon_j, \epsilon_{\psi(i)}) < \delta(\epsilon_{\psi(i)}, \epsilon_{\phi(i)}) = \delta_{\phi(i)-1},$$

since $\delta_{\phi(i)-1} > \delta_j$ for all $\psi(i) \leq j \leq \phi(i) - 1$. This proves the claim in the remaining cases $1 \leq j < \phi(i) - 1$.

Consider the set $\{\phi(2i-1)\}_{i=1}^{n/2}$, which contains $n/2$ distinct elements since $\phi(2i-1) = \max(\pi(2i-1), \pi(2i))$ and these pairs are disjoint. Let $j_1, \ldots, j_{n/2}$ be a permutation of the odd numbers up to $n-1$ such that $\phi(j_1) < \ldots < \phi(j_{n/2})$. By the claim in the previous paragraph, we have $\delta_{\phi(j_1)-1} <$

$\cdots < \delta_{\phi(j_{n/2})-1}$. Consider, for each $r < s$ with $r, s \in \{1, \cdots, n/2\}$, the triple $\{\epsilon_{\psi(j_r)}, \epsilon_{\phi(j_r)}, \epsilon_{\phi(j_s)}\}_<$, which is an edge of the copy of H on S. Since $\psi(j_r) < \phi(j_r) < \phi(j_s)$, by property (b) of function δ and the claim above, $\delta(\epsilon_{\psi(j_r)}, \epsilon_{\phi(j_r)}) = \delta_{\phi(j_r)-1}$ and $\delta(\epsilon_{\phi(j_r)}, \epsilon_{\phi(j_s)}) = \delta_{\phi(j_s)-1}$. Therefore, by the definition of C_1 we must have that $\{\delta_{\phi(j_r)-1}, \delta_{\phi(j_s)-1}\}$ is red. Hence we get a clique of size $n/2$ in our original colouring. But this cannot happen so we have a contradiction. All other cases follow similarly, so we're done. $\quad\square$

This result is closely related to another interesting question: what is the maximum of $r(H)$ over all k-uniform hypergraphs with m edges (we assume here that the hypergraphs we consider do not have isolated vertices)? For graphs, this question was posed by Erdős and Graham [10] who conjectured that the Ramsey number of a complete graph is at least the Ramsey number of every graph with the same number of edges. As noted by Erdős [9], this conjecture implies that there is a constant c such that for all graphs G, $r(G) \leq 2^{c\sqrt{e(G)}}$. The best result in this direction, proven by Alon, Krivelevich and Sudakov [1], is that $r(G) \leq 2^{c\sqrt{e(G)\log e(G)}}$. For hypergraphs, one can naturally ask a question similar to the Erdős-Graham conjecture, i.e, is there a constant $c = c(k)$ such that for every k-uniform hypergraph H, $r(H) \leq t_k(c\sqrt[k]{e(H)})$? The proof of Theorem 6 has the following corollary:

Corollary 2 *There is a positive constant c such that for each positive integer m, there is a 3-uniform hypergraph H with at most m edges such that the 4-colour Ramsey number of H satisfies $r(H, H, H, H) \geq 2^{2^{c\sqrt{m}}}$.*

Indeed the 3-uniform hypergraph H constructed in the proof of Theorem 6 has n vertices and less than n^2 edges, while $r(H, H, H, H) \geq t_3(n/4) \geq t_3(\sqrt{e(H)}/4)$. This corollary demonstrates that the multicolour version of the hypergraph analogue of the Erdős-Graham conjecture is false.

In the other direction, we prove the following theorem:

Theorem 7 *The q-colour Ramsey number of any k-uniform hypergraph H with m edges satisfies*

$$r(H, \cdots, H) \leq t_k(c\sqrt{m})$$

for $k \geq 4$, and

$$r(H, \cdots, H) \leq t_3(c\sqrt{m}\log m)$$

for $k = 3$, where constant c depends only on k and q.

Theorem 7 follows immediately from the remark after the proof of Theorem 4 together with the following lemma.

Lemma 2 *Every k-uniform hypergraph H with m edges has strong chromatic number at most $k\sqrt{m}$.*

Proof: Let H' be the graph on the same vertex set as H with two vertices adjacent if they lie in an edge of H. The strong chromatic number of H is clearly equal to the chromatic number of H'. The number $e(H')$ of edges of H' is at most $\binom{k}{2}m \leq \binom{k\sqrt{m}}{2}$ since each edge of H gives rise to at most $\binom{k}{2}$ edges of H'. To finish the proof, note that the chromatic number χ of any graph with t edges satisfies $\binom{\chi}{2} \leq t$ because in an optimal colouring there should be an edge between any two colour classes. $\quad\square$

5 Conclusion

Throughout this paper we have aimed for simplicity in the exposition. Accordingly, in proving Theorem 2, we have cut some corners to make the proof as pithy as possible. The resulting constant, $c = (2k)^{k-1}$, is doubtless far from best possible, but we believe that this loss is outweighed by the resulting brevity of exposition.

As we noted in the introduction, our Theorem 4 implies that, for $k \geq 4$, there exists a constant $c = c(k)$ such that, for any graph H on n vertices with maximum degree Δ, $r(H) \leq t_k(c\Delta)n$, where the constant c depends only on k. For $k = 3$, however, it only implies that

$$r(H) \leq 2^{2^{c\Delta \log \Delta}} n, \tag{1}$$

which could perhaps be improved a little. It is worth noting also that for $k = 2$, the best known bound, proved by Graham, Rödl and Ruciński [15] using a very different method is

$$r(H) \leq 2^{c\Delta \log^2 \Delta} n. \tag{2}$$

In light of the situation for higher k as well as the lower bound constructions for $k = 2, 3$, the following is a natural question:

Problem 1 *Can the* log *factors in the highest exponent of the upper bounds (1) and (2) be removed?*

This problem is certainly difficult in the $k = 2$ case, but maybe a different extension of the methods of [12] or an appropriate generalisation of the work of Graham, Rödl and Ruciński could resolve the $k = 3$ case.

It also seems likely to us that the lower bound for this problem is essentially the same as the upper bound. So we have the following open problem:

Problem 2 *Is it true that for all k and Δ and sufficiently large n, there exists a k-uniform hypergraph H with maximum degree Δ and n vertices such that $r(H) \geq t_k(c\Delta)n$, where $c > 0$ only depends on k?*

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