Paths and stability number in digraphs

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Abstract

The Gallai-Milgram theorem says that the vertex set of any digraph with stability number k can be partitioned into k directed paths. In 1990, Hahn and Jackson conjectured that this theorem is best possible in the following strong sense. For each positive integer k, there is a digraph D with stability number k such that deleting the vertices of any $k-1$ directed paths in D leaves a digraph with stability number k. In this note, we prove this conjecture.

1 Introduction

The Gallai-Milgram theorem [7] states that the vertex set of any digraph with stability number k can be partitioned into k directed paths. It generalizes Dilworth's theorem [4] that the size of a maximum antichain in a partially ordered set is equal to the minimum number of chains needed to cover it. In 1990, Hahn and Jackson [8] conjectured that this theorem is best possible in the following strong sense. For each positive integer k, there is a digraph D with stability number k such that deleting the vertices of any $k-1$ directed paths in D leaves a digraph with stability number k. Hahn and Jackson used known bounds on Ramsey numbers to verify their conjecture for $k \leq 3$. Recently, Bondy, Buchwalder, and Mercier [3] used lexicographic products of graphs to show that the conjecture holds if $k = 2^a 3^b$ with a and b nonnegative integers. In this short note we prove the conjecture of Hahn and Jackson for all k.

Theorem 1 *For each positive integer k, there is a digraph D with stability number k such that deleting the vertices of any $k-1$ directed paths leaves a digraph with stability number k.*

To prove this theorem we will need some properties of random graphs. As usual, the random graph $G(n,p)$ is a graph on n labeled vertices in which each pair of vertices forms an edge randomly and independently with probability $p = p(n)$.

Lemma 1 *For $k \geq 3$, the random graph $G = G(n,p)$ with $p = 20n^{-2/k}$ and $n \geq 2^{15k^2}$ a multiple of $2k$ has the following properties.*
(a) The expected number of cliques of size $k+1$ in G is at most $20^{\binom{k+1}{2}}$.
(b) With probability more than $\frac{2}{3}$, every induced subgraph of G with $\frac{n}{2k}$ vertices has a clique of size k.

Proof: (a) Each subset of $k+1$ vertices has probability $p^{\binom{k+1}{2}}$ of being a clique. By linearity of expectation, the expected number of cliques of size $k+1$ is

$$\binom{n}{k+1} p^{\binom{k+1}{2}} = \binom{n}{k+1} 20^{\binom{k+1}{2}} n^{-k-1} \leq 20^{\binom{k+1}{2}}.$$

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(b) Let U be a set of $\frac{n}{2k}$ vertices of G. We first give an upper bound on the probability that U has no clique of size k. For each subset $S \subset U$ with $|S| = k$, Let B_S be the event that S forms a clique, and X_S be the indicator random variable for B_S. Since $k \geq 3$, by linearity of expectation, the expected number μ of cliques in U of size k is

$$\mu = \mathbb{E}\left[\sum_S X_S\right] = \binom{\frac{n}{2k}}{k} p^{\binom{k}{2}} \geq \frac{n^k}{2(2k)^k k!} 20^{\binom{k}{2}} n^{1-k} \geq 2n.$$

Let $\Delta = \sum \Pr[B_S \cap B_T]$, where the sum is over all ordered pairs S, T with $|S \cap T| \geq 2$. We have

$$\begin{aligned} \Delta &= \sum_{i=2}^{k-1} \sum_{|S\cap T|=i} \Pr[B_S \cap B_T] = \sum_{i=2}^{k-1} \sum_{|S\cap T|=i} p^{2\binom{k}{2}-\binom{i}{2}} = \sum_{i=2}^{k-1} \binom{n}{i}\binom{n-i}{k-i}\binom{n-k}{k-i} p^{2\binom{k}{2}-\binom{i}{2}} \\ &\leq \sum_{i=2}^{k-1} n^{2k-i} p^{k(k-1)-\binom{i}{2}} \leq 20^{k^2} \sum_{i=2}^{k-1} n^{2-i+i(i-1)/k} \leq k 20^{k^2} n^{2/k}. \end{aligned}$$

Here we used the fact that $i(i-1)/k - i$ for $2 \leq i \leq k-1$ clearly achieves its maximum when $i = 2$ or $i = k-1$.

Using that $k \geq 3$ and $n \geq 2^{15k^2}$, it is easy to check that $\Delta \leq n$. Hence, by Janson's inequality (see, e.g., Theorem 8.11 of [2]) we can bound the probability that U does not contain a clique of size k by $\Pr\left[\wedge_S \bar{B}_S\right] \leq e^{-\mu+\Delta/2} \leq e^{-n}$. By the union bound, the probability that there is a set of $\frac{n}{2k}$ vertices of $G(n,p)$ which does not contain a clique of size k is at most $\binom{n}{\frac{n}{2k}} e^{-n} \leq 2^n e^{-n} < 1/3$. $\square$

The proof of Theorem 1 combines the idea of Hahn and Jackson of partitioning a graph into maximum stable sets and orienting the graph accordingly with Lemma 1 on properties of random graphs.

Proof of Theorem 1. Let $k \geq 3$ and $n \geq 2^{15k^2}$. By Markov's inequality and Lemma 1(a), the probability that $G(n,p)$ with $p = 20n^{-2/k}$ has at most $2 \cdot 20^{\binom{k+1}{2}}$ cliques of size $k+1$ is at least $1/2$. Also, by Lemma 1(b), we have that with probability at least $2/3$ every set of $\frac{n}{2k}$ vertices of this random graph contains a clique of size k. Hence, with positive probability (at least $1/6$) the random graph $G(n,p)$ has both properties. This implies that there is a graph G on n vertices which contains at most $2 \cdot 20^{\binom{k+1}{2}}$ cliques of size $k+1$ and every set of $\frac{n}{2k}$ vertices of G contains a clique of size k. Delete one vertex from each clique of size $k+1$ in G. The resulting graph G' has at least $n - 2 \cdot 20^{\binom{k+1}{2}} \geq 3n/4$ vertices and no cliques of size $k+1$. Next pull out vertex disjoint cliques of size k from G' until the remaining subgraph has no clique of size k, and let $V_1, \ldots, V_t$ be the vertex sets of these disjoint cliques of size k. Since every induced subgraph of G of size at least $\frac{n}{2k}$ contains a clique of size k, then $|V_1 \cup \ldots \cup V_t| \geq \frac{3n}{4} - \frac{n}{2k} \geq \frac{n}{2}$. Define the digraph D on the vertex set $V_1 \cup \ldots \cup V_t$ as follows. The edges of D are the nonedges of G. In particular, all sets V_i are stable sets in D. Moreover, all edges of D between V_i and V_j with $i < j$ are oriented from V_i to V_j. By construction, the stability number of D is equal to the clique number of G', namely k. Also any set of $\frac{n}{2k}$ vertices of D contains a stable set of size k. Note that every directed path in D has at most one vertex in each V_i. Hence, deleting any $k-1$ directed paths in D leaves at least $|D|/k \geq \frac{n}{2k}$ remaining vertices. These remaining vertices contain a stable set of size k, completing the proof. $\square$

Remark. Note that in order to prove Theorem 1, we only needed to find a graph G on n vertices with no clique of size $k+1$ such that every set of $\frac{n}{2k}$ vertices of G contains a clique of size k. The existence of such graphs were first proved by Erdős and Rogers [6], who more generally asked to estimate the minimum t for which there is a graph G on n vertices with no clique of size s such that every set of

t vertices of G contains a clique of size r. Since then a lot of work has been done on this question, see, e.g., [9, 1, 10, 5]. Although most result for this problem used probabilistic arguments, Alon and Krivelevich [1] give an explicit construction of an n-vertex graph G with no clique of size $k+1$, such that every subset of G of size $n^{1-\epsilon_k}$ contains a k-clique. Since we only need a much weaker result to prove the conjecture of Hahn and Jackson, we decided to include its very short and simple proof to keep this note self-contained.

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