On Grids in Topological Graphs

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ABSTRACT

A *topological graph* is a graph drawn in the plane with vertices represented by points and edges as arcs connecting its vertices. A k*-grid* in a topological graph is a pair of subsets of the edge set, each of size k, such that every edge in one subset crosses every edge in the other subset. It is known that for a fixed constant k, every n-vertex topological graph with no k-grid has $O(n)$ edges. We conjecture that this remains true even when: (1) considering grids *with distinct vertices*; or (2) all edges are straight-line segments and the edges within each subset of the grid are required to be *pairwise disjoint*. These conjectures are shown to be true apart from $\log^* n$ and $\log^2 n$ factors, respectively. We also settle the conjectures for some special cases.

1. INTRODUCTION

The *intersection graph* of a set $\mathcal{C}$ of geometric objects has vertex set $\mathcal{C}$ and an edge between every pair of objects with a nonempty intersection. The problems of finding maximum independent set and maximum clique in the intersection graph of geometric objects have received a considerable amount of attention in the literature due to their applications in VLSI design [9], map labeling [1], frequency assignment in cellular networks [12], and elsewhere. Here we study the intersection graph of the edge set of graphs that are drawn in the plane. It is known that if this intersection graph does not contain a large complete bipartite subgraph, then the number of edges in the original graph is small. We show that this remains true even under some very restrictive conditions.

A *topological graph* is a graph drawn in the plane with points as vertices and edges as arcs connecting its vertices. The arcs are allowed to cross, but they may not pass through vertices except for their endpoints. We only consider graphs without parallel edges or self-loops. A topological graph is *simple* if every pair of its edges intersect at most once. If the edges are drawn as straight-line segments, then the graph is *geometric*.

Given a topological graph G, the intersection graph of $E(G)$ has the edge set of G as its vertex set, and an edge between every pair of crossing edges. Note that we consider the edges of G as *open* curves, therefore, edges that intersect only at a common vertex are not adjacent in the intersection graph. A complete bipartite subgraph in the intersection graph of $E(G)$ corresponds to a *grid* structure in G.

Definition 1.1. *A (k,l)-*grid *in a topological graph is a pair of edge subsets E_1, E_2 such that $|E_1| = k$, $|E_2| = l$, and every edge in E_1 crosses every edge in E_2. A k-*grid *is an abbreviation for a (k,k)-grid.*

Theorem 1.2 ([15]). *Given fixed constants $k, l \geq 1$ there exists another constant $c_{k,l}$, such that any topological graph on n vertices with no (k,l)-grid has at most $c_{k,l}n$ edges.*

The proof of Theorem 1.2 in [15] actually guarantees a grid in which all the edges of one of the subsets are adjacent to a common vertex. For two recent and different proofs of Theorem 1.2 see [8] and [7]. Tardos and Tóth [19] extended the result in [15] by showing that there is a constant c_k such that a topological graph on n vertices and at least $c_k n$ edges must contain three subsets of k edges each, such that every pair of edges from different subsets cross, and for two of the subsets all the edges within the subset are adjacent to a common vertex.

Note that according to Definition 1.1 the edges within each subset of the grid are allowed to cross or share a common vertex, as is indeed required in the proofs of [15] and [19]. However, a drawing similar to Figure 1 usually comes to mind when one thinks of a "grid". That is, we would like every pair of edges within each subset of the grid to be *disjoint*, i.e., neither to share a common vertex nor to cross. We say that a (k,l)-grid formed by edge subsets E_1 and E_2 is *natural* if the edges within E_1 are pairwise disjoint, and the edges within E_2 are pairwise disjoint.

Conjecture 1.3. *Given fixed constants $k, l \geq 1$ there exists another constant $c_{k,l}$, such that any simple topological*

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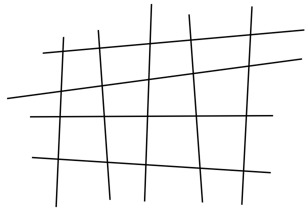


Figure 1: a “natural” grid

graph G on n vertices with no natural (k,l)-grid has at most $c_{k,l}n$ edges.

Note that it is already not trivial to show that an n-vertex geometric graph with no k pairwise disjoint edges has $O(n)$ edges (see [17] and [20]). Moreover, it is an open question whether a simple topological graph on n vertices and no k disjoint edges has $O(n)$ edges (the best upper bound, due to Pach and Tóth [16], is $O(n \log^{4k-8} n)$). Therefore, a proof of Conjecture 1.3 is probably hard to obtain. Here we prove the following bounds for geometric and simple topological graphs with no natural k-grids.

Theorem 1.4.

(i) An n-vertex geometric graph with no natural k-grid has $O(k^2 n \log^2 n)$ edges.

(ii) An n-vertex simple topological graph with no natural k-grid has $O(n \log^{4k-6} n)$ edges.

An n-vertex topological graph with no $(1,1)$-grid is planar and hence has at most $3n-6$ edges, for $n > 2$. We settle Conjecture 1.3 for the first nontrivial case.

Theorem 1.5. *An n-vertex simple topological graph with no natural $(2,1)$-grid has $O(n)$ edges.*

Many extremal problems on geometric graphs become easier for *convex* geometric graphs— geometric graphs whose vertices are in convex position. Indeed, it was already pointed out by Klazar and Marcus [10] that it is not hard to modify the proof of the Marcus-Tardos Theorem [13] and obtain a linear bound for the number of edges in an *ordered* graph that does not contain a certain ordered matching (see [10] for more details). Since crossings in convex geometric graphs are determined by the order of the vertices, this also settles Conjecture 1.3 for convex geometric graphs.

Corollary 1.6. *Given a fixed constant $k \geq 1$ there exists another constant c_k, such that any convex geometric graph on n vertices with no natural k-grid has at most $c_k n$ edges.*

The constant c_k in Corollary 1.6 is huge. Using different techniques, we prove tighter upper bounds for the number of edges in convex geometric graphs avoiding natural $(2,1)$-, $(2,2)$-, or $(k,1)$-grids.

Conjecture 1.3 is clearly false for (not necessarily simple) topological graphs: the complete graph can be drawn as a topological graph in which every pair of edges intersect (at most twice [16]). Therefore, for topological graphs we have to settle for only one of the components of “disjointness”.

Conjecture 1.7. *Given fixed constants $k, l \geq 1$ there exists another constant $c_{k,l}$, such that any topological graph on n vertices with no (k,l)-grid with distinct vertices has at most $c_{k,l}n$ edges.*

This conjecture is shown to be true for $l = 1$.

Theorem 1.8. *An n-vertex topological graph with no $(k,1)$-grid with distinct vertices has $O(n)$ edges.*

For the general case we provide a slightly superlinear upper bound.

Theorem 1.9. *Every n-vertex topological graph with no k-grid with distinct vertices has at most $c_k n \log^* n$ vertices, where $c_k = k^{O(\log\log k)}$ and $\log^*$ is the iterated logarithm function.*

Note that c_k is just barely superpolynomial in k.

Organization. The rest of this paper is organized as follows. We discuss topological graphs with no grids with distinct vertices in Section 2. In Section 3 we prove the bounds for the number of edges in simple topological graphs with no natural grids. Convex geometric graphs are considered in Section 4. We systematically omit floor and ceiling signs whenever they are not crucial for the sake of clarity of presentation. We also do not make any serious attempt to optimize absolute constants in our statements and proofs. All logarithms in this paper are base 2.

2. GRIDS ON DISTINCT VERTICES

In this section we prove Theorems 1.9 and 1.8.

2.1 Topological graphs with no k-grid with distinct vertices

Here we prove Theorem 1.9. We use the following three results from different papers. A graph is a *string graph* if it is an intersection graph of a collection of curves in the plane.

Lemma 2.1 ([5]). *Every string graph with m vertices and ϵm^2 edges contains the complete bipartite graph $K_{t,t}$ as a subgraph with $t \geq \epsilon^{c_1} \frac{m}{\log m}$, where c_1 is an absolute constant.*

The *pair-crossing number* pair-cr(G) of a graph G is the minimum possible number of unordered pairs of crossing edges in a drawing of G. The *bisection width*, denoted by $b(G)$, is defined for every simple graph G with at least two vertices. It is the smallest nonnegative integer such that there is a partition of the vertex set $V = V_1 \dot\cup V_2$ with $\frac{1}{3} \cdot |V| \leq V_i \leq \frac{2}{3} \cdot |V|$ for $i = 1, 2$, and $|E(V_1, V_2)| = b(G)$. We will use the following result of Kolman and Matoušek [11] which relates the pair-crossing number and the bisection width of a graph.

Lemma 2.2 ([11]). *There is an absolute constant c_2 such that if G is a graph with n vertices of degrees $d_1, \ldots, d_n$, then*

$$b(G) \leq c_2 \log n \left(\sqrt{\text{pair-cr}(G)} + \sqrt{\sum_{i=1}^{n} d_i^2} \right).$$

Let G be a topological graph with n vertices and more than $n(\log n)^{c_3 \log h}$ edges. It is shown in [6] that G has h pairwise crossing edges. In [7], it is shown that G has h pairwise crossing edges *with distinct vertices*. This stronger version was needed in the proof of an upper bound on the number of edges in a string graph with a forbidden bipartite subgraph. Here we need an even stronger version for the proof of Theorem 1.9.

THEOREM 2.3. *Let G be a topological graph with n vertices such that the edges of G are colored with each color class forming a matching. If G does not contain h pairwise crossing edges of different colors and with distinct vertices, then the number m of edges of G is at most $n(\log n)^{c_3 \log h}$.*

The proof of Theorem 2.3 is so similar to the proof of the previous weaker versions that we only outline the proof idea, showing details only where they differ from the previous versions in [6] and [7]. The proof is by induction on n and h. If the intersection graph of the m edges is sparse, i.e., there are at most $cm^2/(\log n)^4$ pairs of intersecting edges for some small absolute constant $c > 0$, then we apply Lemma 2.2 and find a partition of the vertices into two subsets with few edges between them. In this case, we are done by the induction hypothesis applied to each of these vertex subsets. If the intersection graph of the edges is dense, i.e., there are more than $cm^2/(\log n)^4$ pairs of edges that intersect, then using Lemma 2.1 we find two large edge subsets E_1, E_2 with $|E_1| = |E_2|$ such that every edge in E_1 intersects every edge in E_2. In [7], it is shown that one can pick $E_1' \subset E_1$ and $E_2' \subset E_2$ with $|E_1'| = |E_2'| \geq |E_2|/8$ such that the vertices that are in edges in E_1' are different from the vertices that are in edges in E_2'. With the next lemma, with $A_i = E_i'$ for $i = 1, 2$, we find subsets $E_1'' \subset E_1'$ and $E_2'' \subset E_2'$ with $|E_1''| = |E_2''|$ such that every edge in E_1'' has different color from every edge in E_2''.

LEMMA 2.4. *Let A_1, A_2 be two disjoint sets such that $|A_1| = |A_2| \geq 2n$. Suppose the elements of $A_1 \cup A_2$ are colored such that no color class has more than $n/2$ elements. Then there are $A_1' \subset A_1$ and $A_2' \subset A_2$ with $|A_1'|, |A_2'| \geq |A_1|/4$ such that every element of A_1' has a different color from every element of A_2'.*

PROOF. Let $c_1, \dots, c_t$ be the colors. In increasing order of j, at step j, if there are at least as many elements in A_1 of color c_j as there are in A_2 of color c_j, then we place the elements of color c_j which are in A_1 in A_1'. If the number of elements of color c_j which are in A_2 is more than the number of elements of color c_j in A_1, then we place all elements of color c_j which are in A_2 in A_2'. We stop this process after j_0 colors if there is i such that $|A_i'| \geq |A_i|/2$. This process stops at some step j_0, since at least half of the elements considered are placed in A_1' or A_2'. Suppose without loss of generality that $|A_1'| \geq |A_1|/2$. For $j_0 < j \leq t$, we also place all elements of color c_j in A_2'. Since at most $|A_1'| \leq |A_1|/2 + n/2$ elements of A_2 are not in A_2', then $|A_2'| \geq |A_2| - |A_1|/2 - n/2 = |A_2|/2 - n/2 \geq |A_2|/4$. By construction, $|A_1'|, |A_2'| \geq |A_1|/4$, and no element of A_1' has the same color as an element in A_2'. □

Not both E_1'' and E_2'' contain $h/2$ pairwise crossing edges of distinct colors and distinct vertices, since otherwise together we would have h pairwise crossing edges of distinct colors and distinct vertices. The induction hypothesis therefore gives an upper bound on the size of E_1'', which completes the proof of Theorem 2.3.

Let $h(k)$ be the minimum h such that if a collection C of h pairwise intersecting curves is such that each of the curves is partitioned into one or two subcurves, then there are k subcurves intersecting k other subcurves, and these $2k$ subcurves come from distinct curves in C. Note that $h(1) = 2$.

LEMMA 2.5. *For $k \geq 2$, we have $h(k) \leq c_4 k \log k$ for some absolute constant c_4.*

PROOF. Let $h = c_4 k \log k$, where $c_4 = 16^{c_1+1}$, where c_1 is the absolute constant in Lemma 2.1. For each curve $\gamma \in C$, randomly pick one of the at most two subcurves to keep. For each pair $\gamma, \gamma' \in C$, there is a probability at least $1/4$ that the subcurve of γ we pick intersects the subcurve of γ' we pick. So the expected number of intersecting pairs of curves is at least $\frac{1}{4}\binom{h}{2}$. So there is a collection C' consisting of one subcurve of the at most two subcurves for each curve such that the number of intersecting pairs of curves in C' is at least $\frac{1}{4}\binom{h}{2}$. Since C' has cardinality h and at least $\frac{1}{4}\binom{h}{2} \geq \frac{1}{16}h^2$ intersecting subcurves, then by Lemma 2.1, the intersection graph of C' contains a complete bipartite graph with parts of size

$$\left(\frac{1}{16}\right)^{c_1} \frac{h}{\log h} \geq k,$$

since we picked c_4 sufficiently large. □

Let $f_k(n)$ denote the maximum number of edges of a topological graph with n vertices and no k-grid with distinct vertices. The remainder of this subsection is devoted toward proving Theorem 1.9, which says that $f_k(n) \leq c_k n \log^* n$. It will be helpful to consider a related function. Let $f_k(n, \Delta)$ denote the maximum number of edges of a topological graph with n vertices, maximum degree at most Δ, and no k-grid with distinct vertices.

We collect several useful lemmas before proving Theorem 1.9. For a graph G and vertex sets A and B, let $e_G(A)$ denote the number of edges with both vertices in A and $e_G(A, B)$ denote the number of pairs $(a, b) \in A \times B$ that are edges of G.

LEMMA 2.6. *There is an absolute constant c such that if $\Delta = (\log n)^{c \log k}$, then*

$$f_k(n) \leq f_k(n, \Delta) + k^{c \log \log k} n.$$

PROOF. Let $G = (V, E)$ be a topological graph with n vertices, $f_k(n)$ edges, and no k-grid with distinct vertices. Partition $V = A \cup B$, where A consists of those vertices with degree more than Δ. We construct a sequence of topological graphs G_i with vertex set A. Let G_0 simply be the induced subgraph of G with vertex set A. Suppose we already have topological graph G_i. If there is a vertex $v \in B$ adjacent to two vertices $a_1, a_2 \in A$ which are not adjacent, then we replace the path of length two with edges (a_1, v) and (v, a_2) by an edge from a_1 to a_2, and let G_{i+1} be the resulting topological graph. We eventually stop at some step j and we have a topological graph G_j on A. Notice that at each step, we delete two edges from B to A and replace it by one edge between two vertices in A. For each vertex $v \in B$, the set A_v of vertices in A adjacent to v after constructing G_j

form a clique in G_j, otherwise v is adjacent to two vertices $a_1, a_2 \in A$ that are not adjacent in G_j, which contradicts that we stopped at step j. Note that G_j has j more edges than the subgraph of G induced by A.

We first provide an upper bound on the number of edges of G_j. Each edge in G_j corresponds to either an edge or a path of length two in G. We assign each edge of G_j a color, where each edge of G_j that is an edge of G gets its own color, and we color the edges of G_j that form a path of length two in G by the middle vertex $v \in B$. Note that by construction this coloring of the edges of G_j has the property that each color class is a matching. So if there are $h(k)$ pairwise intersecting edges in G_j with distinct vertices and distinct colors, then G contains a k-grid with distinct vertices, a contradiction. By Theorem 2.3 and Lemma 2.5, we have

$$\begin{aligned} e_G(A) + j &= e_{G_j}(A) \leq |A|(\log |A|)^{c_3 \log h(k)} \\ &\leq |A|(\log n)^{c_3 \log(c_4 k \log k)} \leq |A|(\log n)^{c_6 \log k} \end{aligned}$$

for some absolute constant c_6.

As discussed above, for each vertex $v \in B$, the set A_v of vertices in A adjacent to v after constructing G_j form a clique in G_j. This clique can not have $h(k)$ pairwise intersecting edges with distinct vertices and distinct colors, otherwise it contains a k-grid with distinct vertices. By Theorem 2.3, we have

$$\binom{|A_v|}{2} \leq |A_v|(\log |A_v|)^{c_3 \log h(k)},$$

so dividing both sides by $|A_v|$ we get

$$|A_v| \leq 2(\log |A_v|)^{c_3 \log h(k)} + 1$$

and finally

$$|A_v| \leq h(k)^{c_7 \log \log h(k)}$$

for some absolute constant c_7. Also using Lemma 2.5, we have

$$|A_v| \leq k^{c_8 \log \log k}$$

for some absolute constant c_8. The number $e_G(A, B)$ of edges of G with one vertex in A and the other vertex in B is

$$2j + \sum_{v \in B} |A_v| \leq 2j + |B| k^{c_8 \log \log k}.$$

Since each vertex in A has degree at least Δ in G, the number $e_G(A) + e_G(A, B)$ of edges in G containing at least one vertex in A is at least $|A|\Delta/2$. So

$$\begin{aligned} |A|\Delta/2 &\leq e_G(A) + e_G(A, B) \\ &\leq e_G(A) + 2j + |B| k^{c_8 \log \log k} \\ &\leq 2|A|(\log n)^{c_6 \log k} + |B| k^{c_8 \log \log k} \\ &\leq 2|A|(\log n)^{c_6 \log k} + n k^{c_8 \log \log k} \end{aligned}$$

If $nk^{c_8 \log \log k} \leq 2|A|(\log n)^{c_6 \log k}$, then we get

$$\Delta \leq 8(\log n)^{c_6 \log k},$$

which contradicts $\Delta = (\log n)^{c \log k}$ with c a sufficiently large constant. So $nk^{c_8 \log \log k} > 2|A|(\log n)^{c_6 \log k}$, and the number of edges in G containing a vertex in A is at most $2k^{c_8 \log \log k} n \leq k^{c \log \log k} n$. Note that every vertex in B in G has degree at most Δ, so $e_G(B) \leq f_k(|B|, \Delta) \leq f_k(n, \Delta)$, where the last inequality follows by adding isolated vertices to B to get a set of n vertices. Therefore, the number $f_k(n)$ of edges of G is at most $f_k(n, \Delta) + k^{c \log \log k} n$. □

Let $d_k(n) = \max_{n' \leq n} f_k(n')/n'$ and $d_k(n, \Delta) = \max_{n' \leq n} f_k(n', \Delta)/n'$. Lemma 2.6 demonstrates that

$$d_k(n) \leq d_k(n, \Delta) + k^{c \log \log k} \tag{1}$$

where $\Delta = (\log n)^{c \log k}$. Note that a triangulated planar graph with n vertices has $3n - 6$ edges, so $d_1(n) = 3 - \frac{6}{n}$ for $n \geq 3$, so $d_k(n) \geq 1$ for $n \geq 3$. By Theorem 2.3, we have

$$d_k(n) \leq (\log n)^{c_3 \log 2k} \tag{2}$$

since a set of $2k$ pairwise crossing edges with distinct vertices in a topological graph contains a k-grid with distinct vertices. We will improve this bound significantly.

Lemma 2.7. *There are absolute constants c_9 and $c_{10} > 0$ such that for each k, n and Δ with $\Delta \geq k$ and $n \geq \Delta^{c_9}$, there is $n_1 \leq 2n/3$ such that*

$$d_k(n_1, \Delta) \geq d_k(n, \Delta)\left(1 - n^{-c_{10}}\right).$$

Proof. Let G be a topological graph with at most n vertices, maximum degree at most Δ, and no k-grid with distinct vertices which has maximum possible average degree among all such topological graphs. Without loss of generality, we may suppose that the number of vertices of G is actually n, and let $m = f_k(n, \Delta)$. Since each vertex has degree at most Δ, then G does not contain a $4k\Delta$-grid. Let the number of crossing pairs of edges of G be ϵm^2, so the underlying graph of G has pair-crossing number at most ϵm^2. By Lemma 2.1, G has an ℓ-grid with $\ell \geq \epsilon^{c_1} \frac{m}{\log m}$. Therefore, we have the inequality $\epsilon^{c_1} \frac{m}{\log m} \leq 4k\Delta$, and we get $\epsilon \leq m^{-\frac{2}{3c_1}}$, where we use $4k\Delta \leq m^{1/6}$ and $\log m \leq m^{1/6}$. By Lemma 2.2, there is an absolute constant c_2 such that if $d_1, \ldots, d_n$ is the degree sequence of G, then

$$\begin{aligned} b(G) &\leq c_2 \log n \left(\sqrt{\text{pair-cr}(G)} + \sqrt{\sum_{i=1}^{n} d_i^2} \right) \\ &\leq c_2 \log n \left(\epsilon^{1/2} m + \Delta \sqrt{n} \right) \\ &\leq c_2 \log n \left(m^{1 - \frac{1}{3c_1}} + \Delta \sqrt{n} \right) \leq m^{1 - c_{10}} \end{aligned}$$

for some constant $c_{10} > 0$.

Therefore, there is a partition $V(G) = V_1 \cup V_2$ such that $|V_1|, |V_2| \leq \frac{2}{3} n$ and $e_G(V_1, V_2) \leq m^{1 - c_{10}}$. Since G has m edges in total, there is $i \in \{1, 2\}$ such that $e_G(V_i) \geq \frac{|V_i|}{n}\left(m - m^{1 - c_{10}}\right)$. Therefore, the subgraph of G induced by V_i has average degree at least a fraction $1 - m^{-c_{10}} \geq 1 - n^{-c_{10}}$ of the average degree of G. Letting $n_1 = |V_i|$, we have $n_1 \leq 2n/3$ and the subgraph of G induced by V_i also has maximum degree at most Δ and does not contain a k-grid with distinct vertices, completing the proof. □

Repeatedly applying Lemma 2.7, we obtain the following lemma.

Lemma 2.8. *Let $\Delta = (\log n)^{c \log k}$ as in Lemma 2.6. There is a constant c' such that $d_k(\Delta^{c'}) \geq (1 - \frac{1}{\Delta}) d_k(n, \Delta) \geq d_k(n, \Delta) - 1$.*

PROOF. Let $n_0 = n$. After one application of Lemma 2.7, we get $d_k(n_1, \Delta) \geq d_k(n, \Delta)\left(1 - n^{-c_{10}}\right)$ for some $n_1 \leq 2n/3$. After j applications of Lemma 2.7, we get $d_k(n_j, \Delta) \geq d_k(n, \Delta) \prod_{i=1}^{j}\left(1 - n_{i-1}^{-c_{10}}\right)$ for some $n_j \leq (2/3)^j n$. Let i_0 be the first value such that $n_{i_0} \leq \Delta^{c'}$, where c' is a sufficiently large constant.

We have

$$\begin{aligned}
d_k(\Delta^{c'}) &\geq d_k(\Delta^{c'}, \Delta) \geq d_k(n_{i_0}, \Delta) \\
&\geq d_k(n, \Delta) \prod_{i=1}^{i_0}\left(1 - n_{i-1}^{-c_{10}}\right) \\
&\geq d_k(n, \Delta)\left(1 - \sum_{i=1}^{i_0} n_{i-1}^{-c_{10}}\right) \\
&\geq d_k(n, \Delta)\left(1 - n_{i_0-1}^{-c_{10}} \sum_{i=0}^{\infty} (2/3)^{c_{10} i}\right) \\
&\geq d_k(n, \Delta)\left(1 - n_{i_0-1}^{-c_{10}} \frac{1}{1-(2/3)^{c_{10}}}\right) \\
&\geq d_k(n, \Delta)\left(1 - (\Delta^{c'})^{c_{10}} \frac{1}{1-(2/3)^{c_{10}}}\right) \\
&\geq d_k(n, \Delta)(1 - \frac{1}{\Delta}).
\end{aligned}$$

By (2), we have $d_k(n) \leq (\log n)^{c_3 \log 2k}$. Since c was chosen sufficiently large in Lemma 2.6, we have $d_k(n, \Delta) \leq d_k(n) \leq \Delta$. Summarizing,

$$d_k(\Delta^{c'}) \geq (1 - \frac{1}{\Delta}) d_k(n, \Delta) \geq d_k(n, \Delta) - 1.$$

□

The last inequality in Lemma 2.8 follows from (2) and the fact that the constant c is chosen sufficiently large.

Combining Lemma 2.6, which gives us inequality (1), and Lemma 2.8 we therefore get that there is an absolute constant C such that

$$d_k((\log n)^{C \log k}) \geq d_k(n) - k^{C \log\log k}.$$

Iterating this inequality until $n \leq k^{2C \log\log k}$, and finally applying the trivial inequality $d_k(n) \leq n/2$ if $n \leq k^{2C \log\log k}$, we get that $d_k(n) = O(k^{2C \log\log k} \log^* n)$, and hence

$$f_k(n) = O(k^{2C \log\log k} n \log^* n),$$

completing the proof of Theorem 1.9. □

2.2 Topological graphs with no $(k, 1)$-grid with distinct vertices

Let $G = (V, E)$ be a topological graph. For every edge $e \in E$ define $X(e)$ to be set of edges in E that cross e and share no common vertex with it. Given a set of edges $E' \subset E$, the *vertex cover number* of E' is the minimum size of a set of vertices $V' \subset V$ such that every edge in E' has at least one of its endpoints in V'. Theorem 1.8 will follow from the next lemma, whose proof is due to Rom Pinchasi [18].

LEMMA 2.9. *Let k be a fixed integer and let $G = (V, E)$ be a topological graph on n vertices, such that for every $e \in E$ the vertex cover number of $X(e)$ is at most k. Then there is a constant c_k, such that G has at most $c_k n$ edges.*

PROOF. We use a standard sampling argument. Let m be the number of edges in G, and let $0 < q < 1$ be a constant. Let G' be the graph obtained from G by taking every vertex of G independently with probability q. Call an edge e' in G' *good* if there is no edge f' in G that crosses e' and shares no vertex with it. Denote by n^* and m^* the expected number of vertices and good edges in G', respectively. Clearly, $n^* = qn$. The probability that an edge e is good is at least $q^2(1-q)^k$, thus $m^* \geq q^2(1-q)^k m$. Observe that two good edges may cross only if they share a vertex. Thus, the good edges form a planar graph by the Hanani-Tutte Theorem (see, e.g., [21]). Therefore, $q^2(1-q)^k m \leq m^* \leq 3n^* = 3qn$, and thus, $m \leq \frac{3}{q(1-q)^k} n$. □

Now let G be an n-vertex topological graph with no $(k, 1)$-grid with distinct vertices. We claim that for every $e \in E(G)$ the vertex cover number of $X(e)$ is at most $2k$. Assume not. Then there is an edge $e \in E(G)$ such that the vertex cover number of $X(e)$ is at least $2k+1$. Pick an edge $(u, v) \in X(e)$ and remove all the other edges in $X(e)$ that are covered by v or u. This can be repeated k times, for otherwise $X(e)$ can be covered by at most $2k$ vertices. The edges we picked along with the edge e form a $(k, 1)$-grid with distinct vertices. This proves Theorem 1.8.

3. NATURAL GRIDS IN GEOMETRIC AND SIMPLE TOPOLOGICAL GRAPHS

In this section we consider natural grids in geometric and simple topological graphs and prove Theorems 1.4 and 1.5.

3.1 Proof of Theorem 1.4

In this section we prove Theorem 1.4, which gives an upper bound on the number of edges of a geometric graph or a simple topological graph without a natural k-grid.

We use the following three results from three different papers. Pach et al. [14] proved the following relationship between the crossing number and the bisection width of a graph.

LEMMA 3.1 ([14]). *If G is a graph with n vertices of degrees $d_1, \ldots, d_n$, then*

$$b(G) \leq 7cr(G)^{1/2} + 2\sqrt{\sum_{i=1}^{n} d_i^2}.$$

Let m be the number of edges in G. Since $\sum_{i=1}^{n} d_i = 2m$ and $d_i \leq n$ for every i, we have

$$b(G) \leq 7\mathrm{cr}(G)^{1/2} + 3\sqrt{mn}. \qquad (3)$$

The following lemma is tight apart from the constant factor.

LEMMA 3.2 ([7]). *For each p there is a constant c_p such that if H is a graph with n vertices, at least $c_p t n$ edges, and is an intersection graph of curves in the plane in which each pair of curves intersect in at most p points, then H contains the complete bipartite graph $K_{t,t}$ as a subgraph.*

We will only need to use the case $p = 1$. The last tool we use is an upper bound on the number of edges of a geometric graph with no k pairwise disjoint edges.

LEMMA 3.3 ([20]). *Any geometric graph with n vertices and no k pairwise disjoint edges has at most $2^9(k-1)^2 n$ edges.*

We now prove Theorem 1.4(i). As the proofs of (i) and (ii) are so similar, we only give the details for (i) and discuss how to modify the proof to obtain (ii).

Proof of Theorem 1.4(i): Let $g_k(n)$ be the maximum number of edges of a geometric graph with n vertices and no natural k-grid. Let G be a geometric graph on n vertices and $m = g_k(n)$ edges with no natural k-grid. Let $c = \max(2^{20}c_1, 144)$, where c_1 is the constant with $p = 1$ from Lemma 3.2. We prove by induction on n that $g_k(n) \leq ck^2 n \log^2 n$. Suppose for contradiction that $g_k(n) > ck^2 n \log^2 n$. Let $\epsilon = 10^{-3} \log^{-2} n$. The proof splits into two cases, depending on whether or not the number of pairs of crossing edges of G is less than ϵm^2.

Case 1: The number of pairs of crossing edges is less than ϵm^2. Then the crossing number of G is less than ϵm^2. By (3), there is a partition $V(G) = V_1 \cup V_2$ with $|V_1|, |V_2| \leq 2n/3$ and the number of edges with one vertex in V_1 and one vertex in V_2 is at most

$$\begin{aligned} b(G) &\leq 7\operatorname{cr}(G)^{1/2} + 3\sqrt{mn} \\ &\leq 7\epsilon^{1/2} m + 3\sqrt{mn} = (7\epsilon^{1/2} + 3\sqrt{n/m})m. \end{aligned}$$

Let $n_1 = |V_1|$ and $n_2 = |V_2|$, so $n = n_1 + n_2$. Then we have

$$\begin{aligned} m = g_k(n) &\leq g_k(|V_1|) + g_k(|V_2|) + b(G) \\ &\leq g_k(n_1) + g_k(n_2) + (7\epsilon^{1/2} + 3\sqrt{n/m})m \\ &\leq ck^2 n_1 \log n_1 + ck^2 n_2 \log n_2 + (7\epsilon^{1/2} + 3\sqrt{n/m})m \\ &\leq ck^2 n \log 2n/3 + (7\epsilon^{1/2} + 3\sqrt{n/m})m \\ &\leq ck^2 n \log n - ck^2 n \log 3/2 + (7\epsilon^{1/2} + 3\sqrt{n/m})m. \end{aligned}$$

This implies

$$\begin{aligned} g_k(n) = m &\leq ck^2 n \frac{\log n - \log 3/2}{1 - 7\epsilon^{1/2} - 3\sqrt{n/m}} \\ &< ck^2 n \log n \frac{1 - (\log 3/2)(\log n)^{-1}}{1 - (\log^{-1} n)/4 - 3c^{-1/2}k^{-1}\log^{-1} n} \\ &< ck^2 n \log n, \end{aligned}$$

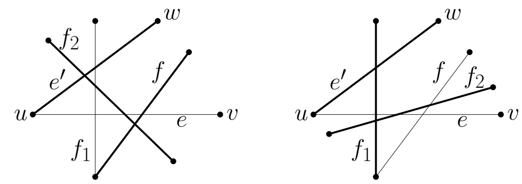
where we use $3c^{-1/2}k^{-1} \leq 1/4$. This completes the proof in this case.

Case 2: The number of pairs of crossing edges is at least ϵm^2. Consider the intersection graph of the edges where two edges are adjacent if they cross. Since this intersection graph has m vertices and at least ϵm^2 edges and each pair of edges intersect at most once, Lemma 3.2 implies it contains a complete bipartite graph with parts of size

$$t \geq \frac{\epsilon m}{c_1} \geq \frac{\log^{-2} nm}{1000c_1} > \frac{c}{1000c_1} k^2 n > 2^9 k^2 n,$$

where c_1 is the constant with $p = 1$ from Lemma 3.2. Therefore, G contains edge subsets E_1, E_2 with $|E_1| = |E_2| = t$ and every edge in E_1 crosses every edge of E_2, i.e., G contains a t-grid. Since $t > 2^9 k^2 n$, Lemma 3.3 implies that E_i contains k disjoint edges for $i = 1, 2$. These two subsets of k disjoint edges cross each other and hence form a natural k-grid, completing the proof. □

To prove Theorem 1.4(ii), essentially the same proof works as above, except replacing the bound $O(k^2 n)$ of Tóth [20] on the number of edges of a geometric graph with no k disjoint edges by the bound $O(n \log^{4k-8} n)$ of Pach and Tóth [16] on the number of edges of a simple topological graph with no k disjoint edges.



(a) f_2 crosses e' (b) f_2 and e' are disjoint

Figure 2: Illustrations for the proof of Lemma 3.4

3.2 Natural $(2,1)$-grids: proof of Theorem 1.5

Let $G = (V, E)$ be a simple topological graph on n vertices without a natural $(2,1)$-grid. For every $e \in E$ assign e the color *red* if $X(e)$ has vertex cover number at most 3, otherwise assign e the color *blue*. It follows from Lemma 2.9 that G has at most $29n$ red edges (by picking $q = 1/4$).

The next lemma is crucial for bounding the number of blue edges. For $F \subseteq E$ denote by $V(F)$ the set of vertices induced by F.

LEMMA 3.4. *Let $e = (u, v)$ be a blue edge, and let $f_1 \in X(e)$. Then if there is an edge $e' = (u, w)$ such that $w \notin V(X(e))$ and e' crosses f_1, then e' crosses every edge $f \in X(e)$.*

PROOF. Assume not. Then there is an edge $f \in X(e)$ such that e' and f do not cross. Note that e' and f must be disjoint since $w \notin V(X(e))$. If f and f_1 are disjoint, then e,f, and f_1 form a natural $(2,1)$-grid. If f and f_1 cross, then e',f, and f_1 form a natural $(2,1)$-grid. Thus, f and f_1 must share a vertex, and $|V(\{f\} \cup \{f_1\})| = 3$. Since e is blue there must be an edge $f_2 \in X(e)$ not sharing an endpoint with f or f_1 (and also not sharing an endpoint with e' since $w \notin V(X(e))$). Therefore f_2 must cross both f and f_1. If f_2 crosses e' then f_2,e', and f form a natural $(2,1)$-grid (see Figure 2(a)). Otherwise, if f_2 and e' are disjoint, then f_2,e', and f_1 form a natural $(2,1)$-grid (see Figure 2(b)). □

Next we remove all the red edges and process the blue edges in some arbitrary order. Let B be the set of the currently unmarked and undeleted blue edges. Initially all the blue edges are in B. Let $e = (u, v)$ be the next edge in B. Delete all the edges that have one endpoint in $V(X(e) \cap B)$ and the other endpoint in $\{u, v\}$. Let E_u be the edges $(u, x) \in B$ such that $x \notin V(X(e) \cap B)$ and there is an edge $e' \in X(e) \cap B$ that crosses (u, x). Similarly, let E_v be the edges $(v, x) \in B$ such that $x \notin V(X(e) \cap B)$ and there is an edge $e' \in X(e) \cap B$ that crosses (v, x). Assume, w.l.o.g., that $|E_u| \geq |E_v|$ and remove the edges E_v. Recall that according to Lemma 3.4, if there is an edge (u, x) such that $x \notin V(X(e))$, and (u, x) crosses some edge in $X(e)$, then (u, x) crosses every edge in $X(e)$.

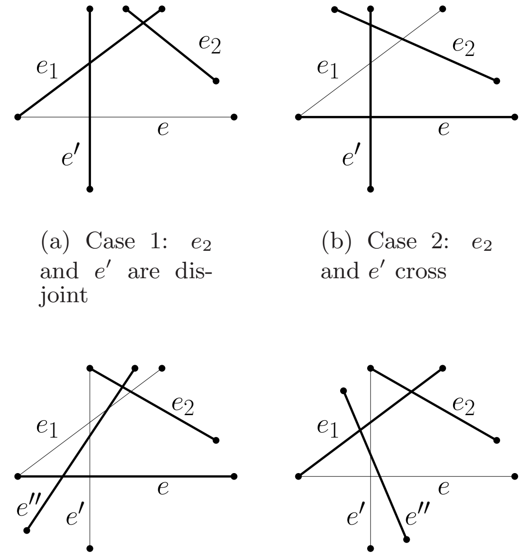
A *thrackle* is a simple topological graph in which every pair of edges meet exactly once, either at a vertex or at a crossing point. It is known that a thrackle on n vertices has at most $3(n-1)/2$ edges [3] and it is a famous open problem (known as Conway's Thrackle Conjecture) to show that the tight bound is n. Set thrackle$(e) = B \cap (\{e\} \cup X(e) \cup \{(u, x) \mid \exists e' \in X(e)$ that crosses $(u, x)\})$. Mark all the blue edges in thrackle(e), and continue to create thrackles as long as there are unmarked blue edges.

LEMMA 3.5. *thrackle(e) is a thrackle.*

PROOF. By definition e meets every other edge in thrackle(e). A pair of edges in $X(e)$ cannot be disjoint, for otherwise they will form a natural $(2, 1)$-grid with e. Finally, by Lemma 3.4 every edge in thrackle(e) of the form (u, x) such that $x \notin X(v)$ must cross all the edges in $X(v)$. □

LEMMA 3.6. *If $e_1 \in$ thrackle(e) and $e_2 \notin$ thrackle(e) then e_1 and e_2 do not cross.*

PROOF. Assume not and let e_1 and e_2 be the first such pair along the process of creating the thrackles. Then, w.l.o.g. e_2 is unmarked when thrackle(e) is created. Clearly $e_1 \neq e$ for otherwise $e_2 \in X(e)$. If $e_1 \in X(e)$ then e_2 does not share a vertex with e, for otherwise it would have been added to thrackle(e) or removed. Thus, e,e_1, and e_2 form a natural $(2, 1)$-grid. Otherwise, e_1 shares a vertex with e and there is an edge $e' \in X(e)$ that crosses e_1. Note that e_2 cannot share a vertex with e, since if it shares the same vertex as e_1 then they cannot cross, and otherwise it would have been removed. There are three possible cases to consider (see Figure 3):
Case 1: e_2 and e' are disjoint. Then e_2,e', and e_1 form a natural $(2, 1)$-grid.
Case 2: e_2 and e' cross. Then e_2,e', and e form a natural $(2, 1)$-grid.
Case 3: e_2 and e' share a vertex. Since e is blue there must an edge $e'' \in X(e)$ that do not share a vertex with e' or e_2. By Lemma 3.4 e'' must cross e_1. If (a) e_2 crosses e'', then e_2,e'', and e form a natural $(2, 1)$-grid. Otherwise, if (b) e_2 and e'' are disjoint then e_2,e'', and e_1 form a natural $(2, 1)$-grid. □



(a) Case 1: e_2 and e' are disjoint

(b) Case 2: e_2 and e' cross

(c) Case 3a: e_2 and e' share a vertex and e_2 crosses e''

(d) Case 3b: e_2 and e' share a vertex and e_2 and e'' are disjoint

Figure 3: Illustrations for the proof of Lemma 3.6

Since any newly created thrackle contains no edges of a previous thrackle, we obtain a partition of the edges that were not deleted into thrackles $t_1, t_2, \ldots, t_j$. Let $t_i =$ thrackle$((u_i, v_i))$ and denote by V_i the vertex set of t_i. Recall that when t_i was created at most $2|V_i|$ edges of the form $(x_i, y_i) \mid x_i \in \{u_i, v_i\} \wedge y_i \in V(X((u_i, v_i)))$ and at most $|V_i|$ edges of the form $(x_i, y_i) \mid x_i \in \{u_i, v_i\} \wedge y_i \notin V(X((u_i, v_i)))$ were deleted. The number of edges in t_i is at most $3|V_i|/2$, thus, it remains to show that $\sum_{i=1}^{j} |V_i| = O(n)$.

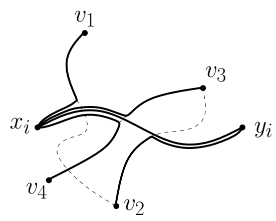
To this end we draw a new graph G' with the same vertex set V. For every thrackle $t_i =$ thrackle$((x_i, y_i))$ we draw a crossing-free tree T_i with $|V_i| - 1$ edges as follows. First, draw the edge from x_i from y_i. Next, for every vertex $v \in V_i \setminus T_i$ pick one edge $e \in t_i$ that has v as one of its endpoints. Follow e from v until it either hits a vertex (necessarily x_i or y_i) or crosses an already drawn edge e'. In the first case draw an edge identical to e. In the second case draw the segment of e from v almost until the crossing point, then continue the edge very close to e' (in one of the directions) until a vertex is reached. See Figure 4(a) for an example.

It follows from Lemma 3.6 and the construction of G' that G' is planar. Note that it is possible for G' to contain parallel edges (see Figure 4(b) for an example). However, it can be shown that they can be eliminated by removing at most half of the edges in G'. Recall that the standard proof using Euler's polyhedral formula that a planar graph on n vertices has at most $3n - 6$ edges (for $n \geq 3$) uses the fact that the graph has no face of size 2 (a 2-*face*). The next lemma will be useful in showing that G' has not too many 2-faces.

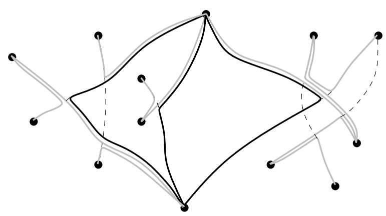
LEMMA 3.7. *Let $t_i =$ thrackle(e) be a thrackle and let p and q be two points on edges of t_i. Then there is a path on edges of t_i between p and q that does not go through any vertex.*

PROOF. It is enough to show that there is a path from p to e. Let e_p be the edge that contains p. If $e_p = e$ then we are done. If e_p crosses e then the segment of e_p between p and the crossing point is the required path. Finally, if e_p does not cross e, then there is an edge $e' \in t_i$ that crosses both e and e_p. The segment of e_p from p to the crossing point of e_p and e' along with the segment of e' from that crossing point to the crossing point of e and e'' create the required path. □

Let t_1, t_2, t_3 be three different thrackles that yield three parallel edges c_1, c_2, c_3 in G' between two vertices u, v. The closed curve $c_1 \cup c_2$ splits the plane into two regions, one containing the interior of c_3. Then this region must contain every vertex in $V_3 \setminus \{u, v\}$. For otherwise, let $w \in V_3 \setminus \{u, v\}$ be a vertex outside that region and let p be some point on c_3. It follows from Lemma 3.7 that there is a path on edges of t_3 between p and w. This path must cross c_1 or c_2 at a point different from u and v, hence there are edges from different thrackles that cross, contradicting Lemma 3.6.



(a) Constructing G'



(b) G' might contain parallel edges

Figure 4: The graph G'

It follows that there are no two adjacent 2-faces (that is, sharing an edge) in G'. Consider the parallel edges between two vertices in G' according to their order around one of the vertices, and remove every other edge. The remaining graph has at least half of the edges of G' and no 2-faces, thus it has at most $3n$ edges. Therefore, G' has at most $6n$ edges, and thus the number of edges in all the thrackles is at most $9n$ and the total number of blue edges is at most $36n$. We conclude that the original graph G has at most $65n$ edges.

4. NATURAL GRIDS IN CONVEX GEOMETRIC GRAPHS

For specific values of k or l we are able to provide tighter bounds in terms of the constant $c_{k,l}$ for the number of edges in convex geometric graphs avoiding natural (k,l)-grids, than the ones guaranteed by Theorems 1.5 and Corollary1.6.

THEOREM 4.1. *An n-vertex convex geometric graph with no natural $(2,1)$-grid has less than $5n$ edges.*

THEOREM 4.2. *An n-vertex convex geometric graph with no natural $(2,2)$-grid has less than $8n$ edges.*

THEOREM 4.3. *A convex geometric graph with $n \geq 3k$ vertices and no natural $(k,1)$-grid has at most $6kn - 12k^2$ edges.*

We mention first some basic notions and facts before moving to the proofs. Let G be a convex geometric graph. We denote by $d_G(v)$ the degree of a vertex v in G, and by $\delta(G)$ the minimum degree in G. For $u, v \in V(G)$, we say that v and u are *consecutive vertices* if they appear next to each other on the convex hull of the vertices of G. For $u, v \in V(G)$ we denote by $R(u,v) \subset V(G)$ the set of vertices from u to v

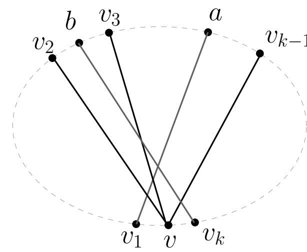


Figure 5: An illustration for the proof of Theorem 4.1

in clockwise order, not including u and v. A convex geometric graph G' is a *geometric minor* of G if G' can be obtained from G by performing a finite number of the following two operations:

1. Vertex deletion.
2. Consecutive vertex contraction, i.e., only consecutive vertices can contract. Recall that the contraction of two vertices x and y, replaces x and y in G with a vertex v, such that v is adjacent to all the neighbors of x and y.

Notice that if two edges e_1 and e_2 cross in G', then they cross in G. Likewise, if e_1 and e_2 are disjoint in G', then they are disjoint in G. Assume that G is a convex geometric graph with n vertices and at least cn edges. Let G' be a minimal geometric-minor of G such that $|E(G')|/|V(G')| \geq c$. Then we can conclude that:

1. $\delta(G') \geq c$ (otherwise vertex deletion can be applied); and
2. every consecutive pair of vertices v and u must have at least $c-1$ common neighbors (otherwise consecutive vertex contraction can by applied).

Proof of Theorem 4.1: Suppose that $|E(G)| \geq 5n$. Let G' be a minimal geometric-minor of G such that $|E(G')|/|V(G')| \geq 5$. Note that $|V(G')| \geq 11$. For a vertex $u \in V(G')$ denote by $u_1, u_2, \ldots$ the neighbors of u in clockwise order. Note that u_1 immediately follows u in clockwise order, since a straight-line segment connecting two consecutive vertices in G cannot be crossed by any edge of G and hence we can assume w.l.o.g. that it is an edge of G. Let $v \in V(G')$ be the vertex such that:

$$|R(v_3, v)| = \min_{u \in V(G')} |R(u_3, u)|$$

Since $\delta(G') \geq 5$, u_3 exists for every u. Since v_1 and v are consecutive vertices they share at least 4 common neighbors. Hence v_1 and v are both adjacent to a vertex $a \in V(G')$, such that $a \notin \{v_2, v_{k-1}, v_k\}$, where $k = d_{G'}(v)$. By minimality of $|R(v_3, v)|$, v_k has at least three neighbors in $R(v_k, v_3)$, See Figure 5. Thus v_k has a neighbor $b \in R(v_k, v_3)$ other than v and v_1. Hence we have a natural $(2,1)$-grid with edges (v, v_{k-1}), (v_1, a), and (v_k, b) in G', and hence in G. □

Proof of Theorem 4.2: Assume that $|E(G)| \geq 8n$. Let G' be a minimal geometric-minor of G with $|E(G')|/|V(G')| \geq$

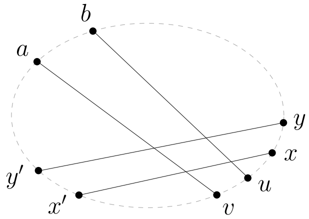
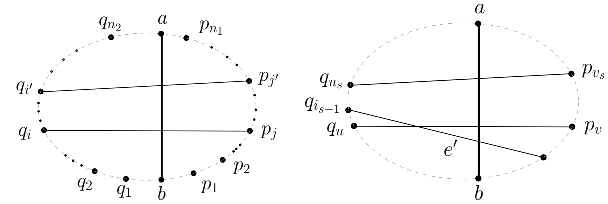


Figure 6: An illustration for the proof of Theorem 4.2



(a) $q_ip_j \prec q_{i'}p_{j'}$

(b) The second case in the proof of Proposition 4.4

Figure 7: Illustrations for the proof of Theorem 4.3

8. Note that $|V(G')| \geq 17$, $\delta(G') \geq 8$, and every pair of consecutive vertices in G' share at least 7 common neighbors. Let (x, x') and (y, y') be a pair of disjoint edges such that:

1. x and y are consecutive vertices with x following y in clockwise order;
2. $|R(x, x')|, |R(y', y)| \geq 2$; and
3. $|R(y', y)|$ is maximized subject to (1) and (2) above.

This is possible since consecutive vertices share at least 7 common neighbors. Now let u, v be the next two vertices after x in clockwise order. Since u and v are consecutive, we know that they share at least 7 common neighbors. Now u and v can have at most 3 common neighbors in $R(v, y') \cup \{y'\}$, since otherwise we would contradict the maximality of $|R(y', y)|$. Hence u and v must have two common neighbors $a, b \in R(y', y)$. See Figure 6. Hence $(x, x'), (y, y'), (u, a), (v, b)$ forms a natural $(2, 2)$-grid in G', and hence in G. □

Proof of Theorem 4.3: Our proof uses the technique from [4]. Let $k \geq 1$ be fixed. We will prove the theorem by induction on the number of vertices n. For $n = 3k$ we need to show that $|E(G)| \leq 6k^2$, however, there are at most $\binom{3k}{2} \leq \frac{9k^2}{2}$ edges. Assume now that the claim is true when the number of vertices is smaller than n and let G be an n-vertex convex geometric graph with no natural $(k, 1)$-grid.

If there is no edge whose endpoints are separated by at least $2k$ vertices along (both arcs of) the boundary of the n-gon, then $|E(G)| \leq 2kn \leq 6kn - 12k^2$ since $n \geq 3k$. So we may assume that there exists such an edge $e = ab$. Assume w.l.o.g. that e is vertical. Let $p_{n_1}, \ldots, p_1$ denote the vertices on the right-hand side of (a, b) and let $q_1, \ldots, q_{n_2}$ denote the vertices on its left-hand side, both in clockwise order. Define a partial order $\prec$ on the set of edges that cross (a, b) as follows: $q_ip_j \prec q_{i'}p_{j'} \Leftrightarrow i < i'$ and $j < j'$ (see Figure 7(a)). We denote by $rank(q_ip_j)$ the largest integer r such that there is a sequence of edges $q_{i_1}p_{j_1} \prec q_{i_2}p_{j_2} \prec \cdots \prec q_{i_r}p_{j_r} = q_ip_j$.

Since G has no natural $(k, 1)$-grid, every edge that crosses ab has rank at most $k - 1$. Next we define a convex geometric graph G_1 with $n_2 + k + 1$ vertices $\{a, p^*_{k-1}, \ldots, p^*_1, b, q_1, \ldots, q_{n_2}\}$ (in clockwise order). Let G_1 be the same as G when restricted to the vertices $\{a, b, q_1, \ldots, q_{n_2}\}$. Then let $q_ip^*_r$ be in $E(G_1)$ if and only if there is an edge $q_ip_j \in E(G)$ whose rank is r. First we will show that if there are t pairwise disjoint edges in G_1 with their *left endpoints* inside an interval (q_i, q_j), then there are t pairwise disjoint edges in G with their *left endpoints* inside the interval (q_i, q_j).

Proposition 4.4. *Let $q_{i_1}p^*_{r_1}, \ldots, q_{i_t}p^*_{r_t}$ be t pairwise disjoint edges in G_1 that cross ab. Then there are t pairwise disjoint edges $q_{u_1}p_{v_1}, \ldots, q_{u_t}p_{v_t}$ such that*

1. *$u_t = i_t$.*
2. *$u_x \geq i_x$ for $x = 1, \ldots, t - 1$.*
3. *$rank(q_{u_x}p_{v_x}) = r_x$, for $x = 1, \ldots, t$.*

Proof. By reverse induction on x. In G we can pick the edge $q_{i_t}p_{v_t}$ that has rank r_t. We know one exists since $q_{i_t}p^*_{r_t}$ exists in G_1. Assume that we have already found the edges $q_{u_x}p_{v_x}$ for $x = t, t-1, \ldots, s > 1$ that satisfy the above. Let q_up_v be an edge of rank r_{s-1} such that $q_up_v \prec q_{u_s}p_{v_s}$. If $u \geq i_{s-1}$, then we can pick q_up_v as the next edge. Otherwise, let e' be an edge of rank r_{s-1} with $q_{i_{s-1}}$ as an endpoint. Since e' and q_up_v have the same rank, they must cross, which implies that $e' \prec q_{u_s}p_{v_s}$ and so we can pick e' as the next edge. See Figure 7(b). □

Proposition 4.5. *G_1 does not contain a natural $(k, 1)$-grid.*

Proof. Assume that G_1 contains a natural $(k, 1)$-grid. Then by considering the possible edges involved in such a grid and using Proposition 4.4 above, one concludes that there is a natural $(k, 1)$-grid in G, which is a contradiction. □

We also define a convex geometric graph G_2 with n_1+k+1 vertices $\{a, p_{n_1}, \ldots, p_1, b, q^*_1, \ldots, q^*_{k-1}\}$ (in clockwise order). Let G_2 be the same as G when restricted to the vertices $\{a, p_{n_1}, \ldots, p_1, b\}$. Then let (q^*_r, p_j) be in $E(G_2)$ if and only if there is an edge $(q_i, p_j) \in E(G)$ whose rank is r. By the same arguments G_2 does not contain a natural $(k, 1)$-grid. Let E_r denote the edges in G with rank r, $1 \leq r \leq k - 1$.

Proposition 4.6. $|E_r| \leq d_{G_1}(p^*_r) + d_{G_2}(q^*_r) - 1$.

Proof. The edges in E_r cannot form a cycle. Indeed, consider a path $q_{i_1}p_{j_1}, q_{i_2}p_{j_1}, q_{i_2}p_{j_2}, \ldots$ and assume w.l.o.g. that $i_1 < i_2$. Then $j_2 < j_1$ for otherwise $q_{i_1}p_{j_1}$ and $q_{i_2}p_{j_2}$ are disjoint. Similarly, we have $i_l > i_{l-1}$ and $j_l < j_{l-1}$, for any $l > 1$, therefore the path can not form a cycle. Since there are $d_{G_1}(p^*_r) + d_{G_2}(q^*_r)$ vertices that are endpoints of edges in E_r, the claim follows. □

Denote by E'_1 the edges in G_1 that do not cross ab and by E'_2 the edges in G_2 that do not cross ab (note that $ab \in E'_i$,

$i = 1, 2$). Recall that ab has at least $2k$ vertices on each of its sides, therefore, $|V(G_1)|, |V(G_2)| \geq 3k$. Then:

$$\begin{aligned}
|E(G)| &= |E'_1| + |E'_2| - 1 + \sum_{r=1}^{k-1} |E_r| \\
&= |E'_1| + |E'_2| - 1 + \sum_{r=1}^{k-1} \Big(d_{G_1}(p_r^*) + d_{G_2}(q_r^*) - 1 \Big) \\
&= |E(G_1)| + |E(G_2)| - k \\
&\overset{\text{ind hyp}}{\leq} (6k(n_1 + k + 1) - 12k^2) \\
&\quad + (6k(n_2 + k + 1) - 12k^2) - k \\
&= 6kn - 12k^2 - k \leq 6kn - 12k^2
\end{aligned}$$

This completes the proof of Theorem 4.3. □

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