Rainbow Solutions to the Sidon Equation

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Abstract

We prove that for every 4-coloring of $\{1, 2, \ldots, n\}$, with each color class having cardinality more than $\frac{n+1}{6}$, there exists a solution of the equation $x + y = z + w$ with x, y, z and w belonging to different color classes. The lower bound on a color class cardinality is tight.

1 Introduction

Let $\mathbb{N}$ denote the set of positive integers, and for $i, j \in \mathbb{N}$, $i \leq j$, let $[i, j]$ denote the set $\{i, i+1, \ldots, j\}$ (with $[n]$ abbreviating $[1, n]$ as usual). One of the earliest results in Ramsey theory [8] is Schur's theorem (1916) [17]: for every $k \in \mathbb{N}$ and sufficiently large $n \in N$, every k-coloring of $[n]$ contains a monochromatic solution of the equation $x + y = z$. Another classical result in combinatorial number theory is due to van der Waerden (1927) [21]: for all $m, k \in N$, there is an integer $n_0 = n_0(m, k)$, such that every k-coloring of $[n]$, $n \geq n_0$, contains a monochromatic m-term arithmetic progression (abbreviated as $AP(m)$ throghout). This statement was further generalized to sets of positive upper density in the celebrated work of Szemerédi [19] (see also [20]). Canonical versions of van der Waerden's theorem were discovered by Erdős and others [7].

More than seven decades after Schur's result, Alekseev and Savchev [1] considered what Bill Sands calls an *un-Schur* problem [9]. They proved that for every equinumerous 3-coloring of $[3n]$ (i.e., a coloring in which different color classes have the same cardinality), the equation $x + y = z$ has a solution with x, y and z belonging to different color classes. Such solutions will be called *rainbow* solutions. Esther Klein and George Szekeres asked whether the condition of equal cardinalities for three color classes can be weakened [18]. Indeed, Schönheim [16] proved that for every 3-coloring of $[n]$, such that every color class has cardinality greater than $n/4$, the equation $x + y = z$ has rainbow solutions. Moreover, he showed that $n/4$ is optimal.

Inspired by the *un-Schur* problem, Jungić et al. [10] sought a rainbow counterpart of van der Waerden's theorem. Namely, given positive integers m and k, what conditions on k-colorings of $[n]$ guarantee the existence of an $AP(m)$, all of whose elements have distinct colors? If every integer in $[n]$ is colored by the largest power of three that divides it, then one immediately obtains a k-coloring of $[n]$ with $k \leq \lfloor \log_3 n + 1 \rfloor$ and without a rainbow $AP(3)$. So, while Szemerédi's theorem states that a large cardinality in only one color class ensures the existence of a monochromatic $AP(m)$, one needs *all* color classes to be "large" to force a rainbow $AP(m)$. In [10], it was proved that every 3-coloring of $\mathbb{N}$ with the upper density of each color class greater than $1/6$ yields a rainbow $AP(3)$. Using some tools from additive number theory, they obtained similar (and stronger) results

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for 3-colorings of $\mathbb{Z}_n$ and $\mathbb{Z}_p$, some of which were recently extended by Conlon [5]. The more difficult *interval* case was studied in [11], where it was shown that every equinumerous 3-coloring of $[3n]$ contains a rainbow $AP(3)$, that is, a rainbow solution to the equation $x+y=2z$. Finally, Axenovich and Fon-Der-Flaass [2] cleverly combined the previous methods with some additional ideas to obtain the following theorem, conjectured in [10].

Theorem 1 *For every $n \geq 3$, every partition of $[n]$ into three color classes $\mathcal{R}$, $\mathcal{B}$, and $\mathcal{G}$ with $min\{|\mathcal{R}|,|\mathcal{B}|,|\mathcal{G}|\} > r(n)$, where*

$$r(n) := \begin{cases} \lfloor (n+2)/6 \rfloor & \text{if } n \not\equiv 2 \pmod 6 \\ (n+4)/6 & \text{if } n \equiv 2 \pmod 6, \end{cases} \tag{1}$$

contains a rainbow $AP(3)$.

The colorings

$$c(i) := \begin{cases} R & \text{if } i \equiv 1 \pmod 6 \\ B & \text{if } i \equiv 4 \pmod 6 \\ G & \text{otherwise} \end{cases} \quad \text{and} \quad \bar{c}(i) := \begin{cases} R & \text{if } i \leq \frac{n+1}{3} \text{ and } i \text{ is odd} \\ B & \text{if } i \geq \frac{2n+2}{3} \text{ and } i \text{ is even} \\ G & \text{otherwise} \end{cases}$$

show that Theorem 1 is the best possible for the cases $n \not\equiv 2 \pmod 6$ and $n \equiv 2 \pmod 6$, respectively. It is interesting to note that similar statements about the existence of rainbow $AP(k)$ in k-colorings of $[n]$, $k \geq 4$, do not hold [2, 6]. For example, the equinumerous 4-coloring $\lambda : [n] \mapsto \{R, B, G, Y\}$

$$\lambda(i) := \begin{cases} R & \text{if } i \equiv 1 \pmod 4 \text{ and } i < 4m; \text{ or if } i \equiv 3 \pmod 4 \text{ and } i > 4m \\ B & \text{if } i \equiv 2 \pmod 4 \text{ and } i < 4m; \text{ or if } i \equiv 0 \pmod 4 \text{ and } i > 4m \\ G & \text{if } i \equiv 3 \pmod 4 \text{ and } i < 4m; \text{ or if } i \equiv 0 \pmod 4 \text{ and } i \leq 4m \\ Y & \text{if } i \equiv 1 \pmod 4 \text{ and } i > 4m; \text{ or if } i \equiv 2 \pmod 4 \text{ and } i > 4m, \end{cases}$$

for every $i \in [n]$, $n = 8m$ $(m \in \mathbb{N})$, contains no rainbow $AP(4)$.

There are many directions and generalizations one can consider, such as searching for rainbow counterparts of other classical theorems in Ramsey theory [8, 12], increasing the number of colors or the length of a rainbow AP, or proving the existence of more than one rainbow AP. Some positive and negative results in these directions were obtained in [3, 4, 10].

In this paper, we study one such direction and consider the existence of rainbow solutions to other linear equations, imitating Rado's theorem about the monochromatic analogue. Rado [15] called a rational matrix A (or a system $Ax = 0$) *k-partition regular* if there exists an n for which every k-coloring of $[n]$ has a monochromatic solution to the system of linear homogeneous equations $Ax = 0$. Furthermore, A is called *partition regular* if it is k-partition regular for all k. Rado's "columns condition" completely determines the matrices (or systems) which are partition regular. A special case of this theorem states that a single linear homogeneous equation $\sum_{i=1}^{m} a_i = 0$, $a_i \in \mathbb{Z}$ is partition regular if and only if some nonempty subset of the a_is sums to zero.

In particular, "the Sidon equation" $x + y = z + w$, a classical object in additive number theory [13, 14] is partition regular. In this note, we prove a rainbow analogue of this result.

Theorem 2 *For every $n \geq 4$, every partition of $[n]$ into four color classes $\mathcal{R}$, $\mathcal{B}$, $\mathcal{G}$, and $\mathcal{Y}$, with $min\{|\mathcal{R}|,|\mathcal{B}|,|\mathcal{G}|,|\mathcal{Y}|\} > \frac{n+1}{6}$, contains a rainbow solution of $x + y = z + w$. Moreover, this result is tight.*

One should contrast Theorem 2 with the aforementioned result of Conlon et al. [6], which states that there nevertheless exist equinumerous 4-colorings of $[n]$ with no rainbow $AP(4)$, i.e., with no rainbow solution of the system $x + y = z + w$, $x + w = 2z$.

2 Proof of Theorem 2

We prove Theorem 2 for $n \geq 5$. Given partition of $[n]$ into four color classes $\mathcal{R}$, $\mathcal{B}$, $\mathcal{G}$, and $\mathcal{Y}$, with $\min\{|\mathcal{R}|, |\mathcal{B}|, |\mathcal{G}|, |\mathcal{Y}|\} > \frac{n+1}{6}$, let $c : [n] \mapsto \{R, G, B, Y\}$ be the corresponding coloring of $[n]$, i.e., $\mathcal{R} = |[n] \cap \{i : c(i) = R\}|$, and similarly for $\mathcal{B}$, $\mathcal{G}$, and $\mathcal{Y}$. Suppose that there is no rainbow solution of the equation $x + y = z + w$.

We say that there is a *string* $\mathbf{s} = c_1 c_2 \ldots c_m \in \{R, G, B, Y\}^m$ at a position i if $c(i) = c_1$, $c(i+1) = c_2$, $\ldots$, $c(i+m-1) = c_m$. We say that there is a string $\mathbf{s}$ in the coloring c if there is $\mathbf{s}$ at some position i. We call a string *bichromatic* if it contains exactly two colors. A bichromatic string is *complete* if it cannot be extended (on either side) and still be bichromatic. Notice that since each color is used at least once, there are at least three complete bichromatic strings.

Since c does not contain a rainbow solution of $x + y = z + w$, then there are no integers a, b, d, such that a, $b + d$, b, $a + d$ form a rainbow solution. In what follows, this observation will be denoted as the *Q-property.*

A particular color is called *dominant* if every bichromatic string contains that color. Clearly, if such a color exists, it will be unique. The first step in our proof is to establish the following claim.

Lemma 1 *c contains a dominant color.*

Proof: Consider the first two complete bichromatic strings, i.e., those with the least initial position. They share a common color. Without loss of generality, assume that the first bichromatic string contains colors R and B, and the second bichromatic string contains colors R and Y. In particular, R and B, as well as R and Y, occur next to each other. There exists at least one element of $[n]$ colored by G, and this element is contained in a bichromatic string. If the other color in the string is B (Y), then G and B (Y) appear next to each other within this string. Since R and Y (B) are consecutive, we have a contradiction by the Q-property with $d = 1$. Therefore, every bichromatic string that contains G also contains R.

Finally, suppose there is a bichromatic string with colors B and Y. Then B and Y appear next to each other, and since G and R appear next to each other as well, we obtain a contradiction with the Q-property for $d = 1$. We conclude that every bichromatic string contains R, and, therefore, R is the dominant color. □

Now, we can assume that R is the dominant color in c. Let d be the minimum distance between two differently colored non-red integers, that is

$$d = \min\{|x - y| : c(x) \neq c(y) \text{ and } x, y \notin \mathcal{R}\}.$$

Note that because R is the dominant color, we have $d \geq 2$. Without loss of generality, assume that there exist two elements of $[n]$, distance d apart, that are colored by B and Y respectively. By the Q-property, there do not exist two elements of $[n]$, distance d apart, that are colored by R and G respectively. Next, we prove that every complete bichromatic string with colors R and G (B) has a special structure.

Lemma 2 *Let $X \in \{G, B\}$. Every complete bichromatic string with colors R and X is d-periodic with exactly one element colored by X within every substring of length d.*

Proof: Consider a complete bichromatic string $\mathbf{s}$ of length m at a position i, with colors R and G. The underlying interval $I = [i, i + m - 1]$ is the disjoint union of I_k, $0 \leq k \leq d - 1$, where

$I_k = \{j \in [i, i+m-1] \mid j \equiv k (\text{mod } d)\}$. By the Q-property, for every $0 \le k \le d-1$, either all elements of I_k are colored by G or all elements of I_k are colored by R.

Assume that $i \ne 1$. The case $i+m-1 \ne n$ is symmetric and handled similarly. Let g denote the smallest element of I colored by G. If $g-d \ge i$ then $\{g, g-d\} \subseteq I_k$ for some $k \in \{0, 1, \ldots, d-1\}$. So, $c(g-d) = G$, which contradicts our choice of g. Thus, $g-d < i$. Since $\mathbf{s}$ is complete, $c(i-1) \in \{B, Y\}$ and $g-(i-1) \le d$. Therefore, $g-d = i-1$. Now, since $c(g-d) \in \{B, Y\}$, $c(g) = G$ and all the integers between $g-d$ and g are colored by R, we conclude that all the elements of I_k, for $k \equiv g (\text{mod } d)$, are colored by G, while for all other values of $k \in \{1, \ldots, d\}$, all the elements of I_k are colored by R.

Hence, from the above argument we see that every complete bichromatic string with colors R and G has the following structure: it is d-periodic with exactly one element colored by G within every substring of length d. Moreover, since $c(g-d) \in \{B, Y\}$ and $g-d = i-1$, it follows that we can assume, without loss of generality, that there exist two elements of $[n]$, distance d apart, that are colored by G and, say, Y, respectively. The previous argument then implies that every complete bichromatic string with colors R and B is d-periodic with exactly one element colored by B within every substring of length d. $\square$

In particular, since R is the dominant color, we obtain:

Corollary 1 *Strings GG and BB do not appear in c.*

Now, the following claim is clear:

Lemma 3 *String YY appears in c.*

Proof: Suppose that YY does not appear in c. Then, by Corollary 1, at least one in every pair of consecutive integers in $[n]$ would be colored by R. Therefore, $|\mathcal{R}| \ge \lfloor \frac{n}{2} \rfloor \ge \frac{n-1}{2}$, and $3\min\{|\mathcal{Y}|, |\mathcal{G}|, |\mathcal{B}|\} \le |\mathcal{Y}| + |\mathcal{G}| + |\mathcal{B}| = n - |\mathcal{R}| \le \frac{n+1}{2}$. So $\min\{|\mathcal{Y}|, |\mathcal{G}|, |\mathcal{B}|, |\mathcal{R}|\} \le \frac{n+1}{6}$, which contradicts our assumption. $\square$

Lemma 4 $d = 2$.

Proof: Indeed, suppose that $d \ge 3$. By Lemma 2, we have $|\mathcal{R}| \ge (d-1)(|\mathcal{G}| + |\mathcal{B}| - 1)$. Then for $n \ge 5$, $n = |\mathcal{R}| + |\mathcal{B}| + |\mathcal{G}| + |\mathcal{Y}| \ge (d-1)(\frac{n+2}{6} + \frac{n+2}{6} - 1) + \frac{n+2}{2} > n$, which is a contradiction. $\square$

Lemma 2 and Lemma 4 imply the following claim:

Corollary 2 *Let $X \in \{G, B\}$. There exist two integers in $[n]$, with difference 2, that are colored by X and Y, respectively. Furthermore, elements of every bichromatic string with colors R and X alternate in color.*

Lemma 5 *Strings BRG and GRB do not appear in c.*

Proof: Since (by Lemma 3) there is a string of at least two consecutive Ys and since R is the dominant color in c, there are two integers in $[n]$, distance two apart, that are colored by Y and R. The claim now follows from the Q-property. $\square$

Lemma 6 *At least one of the strings GRG and BRB appears in c.*

Proof: Suppose that there is no GRG nor BRB in c. Let us consider four consecutive integers i, $i+1$, $i+2$, $i+3$ in $[n]$. If $c(i+1) = G$, then $c(i) = c(i+2) = R$, by the dominance of color R and Corollary 1. Furthermore, $c(i+3) \in \{R, Y\}$, by Lemma 5. If $c(i) = G$, then $c(i+1) = R$, by the dominance of color R and Corollary 1. Since $c(i) = G$ and $c(i+1) = R$ belong to a bichromatic string with colors R and G (which alternates in color by Corollary 2), if we assume that GRG does not appear in c, then $c(i+2) = Y$, by Lemma 5. It follows that $c(i+3) \in \{R, Y\}$.

Therefore, at most one integer in every string of length four can be colored by B or G. We obtain $|\mathcal{G}| + |\mathcal{B}| \leq \lceil \frac{n}{4} \rceil$, and for $n \geq 5$, $\min\{|\mathcal{R}|, |\mathcal{G}|, |\mathcal{B}|, |\mathcal{Y}|\} \leq \min\{|\mathcal{G}|, |\mathcal{B}|\} \leq \frac{n+3}{8} \leq \frac{n+1}{6}$. This violates our condition on the minimum of color class cardinalities. □

By Lemma 6 we can assume that GRG appears in c. By Lemma 3 there exists p, the smallest positive integer with the property that there is $i \in [n]$ such that $c(i) \in \{G, B\}$ and at least one of the following is true:

(a) $c(i+p) = c(i+p+1) = Y$; $c(i+p-1) = R$; $c(i+j) \in \{R, Y\}$ for all $1 \leq j \leq p-1$ with $R \in \{c(i+j), c(i+j+1)\}$ for $1 \leq j \leq p-2$;

(b) $c(i-p) = c(i-p-1) = Y$; $c(i-p+1) = R$; $c(i-j) \in \{R, Y\}$ for all $1 \leq j \leq p-1$ with $R \in \{c(i-j), c(i-j-1)\}$ for $1 \leq j \leq p-2$.

Next, suppose that there is $i \in [n]$ such that $c(i) = B$ and that, say, (a) is true. Let m be such that $c(i+p+j) = Y$ for all $1 \leq j \leq m$ and, if $i+p+m+1 \in [n]$, $c(i+p+m+1) = R$. Let $k \in [n]$ be such that $c(k) = c(k+2) = G$. We note that $k \notin [i, i+p+m+1]$. Suppose $k > i+p+m+1$. If $c(k-p) = R$, then i, $i+p$, $k-p$, k contradict the Q-property. If $c(k-p) \in \{B, G, Y\}$, then $c(k-p+1) = R$ and i, $i+p+1$, $k-p+1$, $k+2$ contradict the Q-property.

Now, suppose that there is no $i \in [n]$ such that $c(i) = B$ and that either (a) or (b) is true. Thus, there is $i \in [n]$ such that $c(i) = G$ and that, say, (a) is true. Let m be as above and let ℓ be an element from $\mathcal{B}$ such that between ℓ and $[i, i+p+m+1]$ there are no other elements from $\mathcal{B}$. Suppose this time that $\ell < i$. If $c(\ell+p) = R$, then ℓ, $\ell+p$, i, $i+p$ contradict the Q-property. If $c(\ell+p) \in \{G, Y\}$, then $c(\ell+p+1) = R$ and ℓ, $\ell+p+1$, i, $i+p+1$ contradict the Q-property (if $c(\ell+p) = Y$, then $c(\ell+p+1) = R$, because of the minimality of p and the assumption from the beginning of this paragraph).

In order to finish the proof of Theorem 2, we present a 4-coloring of $[n]$ with the minimum size of a color class equal to $\lfloor \frac{n+1}{6} \rfloor$ and no rainbow solution of $x + y = z + w$:

$$c(i) := \begin{cases} B & \text{if } i \equiv 1 \pmod 6 \\ G & \text{if } i \equiv 3 \pmod 6 \\ Y & \text{if } i \equiv 5 \pmod 6 \\ R & \text{otherwise} \end{cases}$$

3 Concluding remarks

It is curious to note that the minimal "density" for the color classes is $\frac{1}{6}$ in Theorem 2, as well as in Theorem 1. It is also interesting to note that a dominant color exists when one studies the existence of rainbow solutions to equations $x + y = 2z$ or $x + y = z$ in the 3-colorings of $[n]$ [2, 10, 11]. For what other systems of equations does a rainbow-free coloring, under certain cardinality constraints, must have a dominant color?

The question of *rainbow partition regularity* is an interesting one. It would be exciting to provide a complete rainbow analogue of Rado's theorem (which classified the partition regular matrices [15]). Theorem 2 is a small step in this direction.

We say a vector is *rainbow* if every entry of the vector is colored differently. A matrix A with rational entries is called *rainbow partition k-regular* if for all n and every equinumerous k-coloring of $[kn]$ there exists a rainbow vector x such that $Ax = 0$. We say that A is *rainbow regular* if there exists k_1 such that A is rainbow partition k-regular for all $k \geq k_1$. For example, Theorem 2 shows that the following matrix is rainbow partition 4-regular:

$$A = \begin{pmatrix} 1 & 1 & -1 & -1 \end{pmatrix}.$$

We let the *rainbow number* of A, denoted by $r(A)$, be the least k for which A is rainbow partition k-regular. It is not difficult to see that every $1 \times n$ matrix A with nonzero entries is rainbow partition regular if and only if not all the entries in A are of the same sign. It would be interesting to study the rainbow number $r(A)$. Furthermore, we somewhat boldly conjecture the following characterization of rainbow regularity.

Conjecture 1 *Matrix A with integer entries is rainbow regular if and only if the rows of A are linearly independent and there exists a vector u with positive integer entries such that $Au = 0$.*

Jungić et al. [10] prove that for every $k \geq 3$, $\lfloor \frac{k^2}{4} \rfloor < r(A) \leq \frac{k(k-1)^2}{2}$, where A is the following $(k-1) \times (k+1)$ matrix:

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & -2 & 1 \end{pmatrix}.$$

Acknowledgments

We thank anonymous referees for helpful comments and a simplification of our proof.

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