The minimum degree of Ramsey-minimal graphs

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ABSTRACT

We write $H \to G$ if every 2-coloring of the edges of graph H contains a monochromatic copy of graph G. A graph H is G-minimal if $H \to G$, but for every proper subgraph H' of H, $H' \not\to G$. We define $s(G)$ to be the minimum s such that there exists a G-minimal graph with a vertex of degree s. We prove that $s(K_k) = (k-1)^2$ and $s(K_{a,b}) = 2\min(a,b) - 1$. We also pose several related open problems.

Keywords: Ramsey, critical, minimal, graph

1. INTRODUCTION

In this paper, we only consider finite simple graphs. The complete graph and the cycle on n vertices are denoted K_n and C_n, respectively.

We write $H \to (G; r)$ if every r-coloring of the edges of H contains a monochromatic copy of G, and $H \to G$ if $H \to (G; 2)$. A graph H is *G-minimal* if $H \to G$, but for every proper subgraph H' of H, $H' \not\to G$. For example, K_6 is K_3-minimal because $K_6 \to K_3$, and no proper subgraph $H' \subset K_6$ has the property that $H' \to K_3$.

The *Ramsey number* $R(G)$ is the minimum positive integer n such that $K_n \to G$. In 1967, Erdős and Hajnal [5] asked whether there exists a K_6-free graph H such that

$H \to K_3$. Ron Graham [10] answered this question by showing that $K_8 - C_5$ (the graph formed by taking the edges of a C_5 out of a K_8) is K_3-minimal. The problem of Erdős and Hajnal opened up the area of research of finding graphs H such that $H \to G$ for a given graph G.

It is easy to prove by induction that a graph H satisfies $H \to G$ if and only if there exists a subgraph H' of H such that H' is G-minimal.

Given graphs $G_1, G_2, \ldots, G_n$, their *product* $G_1 \otimes G_2 \otimes \cdots \otimes G_n$ consists of vertex disjoint copies of $G_1, G_2, \ldots, G_n$ and all possible edges between the vertices of G_i and G_j with $i \neq j$. Galluccio et al. [9] and Szabó [18] proved that for all positive integers k, $C_{2k+1} \otimes K_3$ is K_3-minimal. For $k = 1$ and 2, these graphs are K_6 and Graham's graph $K_8 - C_5$, respectively.

Noting that the K_3-minimal graphs of the form $C_{2k+1} \otimes K_3$ all have minimum degree 5, it is natural to investigate whether there are K_3-minimal graphs of minimum degree less than 5. This question motivates the following definition.

Definition. Let $s(G)$ be the least nonnegative integer s such that there exists a G-minimal graph with a vertex of degree s.

In Section 3, we first show that $2\delta(G) - 1 \leq s(G) \leq R(G) - 1$, where $\delta(G)$ denotes the minimum degree of G. We then prove $s(K_k) = (k-1)^2$ and $s(K_{a,b}) = 2\min(a,b) - 1$. An important part of the proof that $s(K_k) = (k-1)^2$ relies on a natural generalization of a famous theorem of Jaroslav Nešetřil and Vojtěch Rödl, which we prove in Section 2.

In the Conclusion, we consider associated Ramsey numbers and multicolored generalizations.

2. GENERALIZATION OF A THEOREM OF NEŠETŘIL & RÖDL

The *clique number* $\omega(G)$ of a graph G is the number of vertices in the largest complete subgraph of G. We write $F \xrightarrow{\text{ind}} (G; r)$ if every r-coloring of the edges of graph F contains a monochromatic induced copy of G.

Theorem 1 (Nešetřil & Rödl, [13]). For every positive integer r and graph G, there exists a graph F with $\omega(F) = \omega(G)$ and $F \xrightarrow{\text{ind}} (G; r)$.

While Theorem 1 only considers edge colorings, our generalization of Theorem 1 will simultaneously consider both edge and vertex colorings.

For a graph G, let $\mathcal{F}(G, r)$ denote the family of graphs F with $\omega(F) = \omega(G)$ and $F \xrightarrow{\text{ind}} (G; r)$. Theorem 1 is equivalent to $\mathcal{F}(G, r)$ being nonempty for every graph G and positive integer r. Let $\mathcal{F}(G, k, r)$ denote the family of graphs F with $\omega(F) = \omega(G)$, and for every k-coloring of the vertices of F and every r-coloring of the edges of F, there exists an induced copy of G with all its edges the same color and all its vertices the same color.

Theorem 2. For every non-bipartite graph G without isolated vertices and for all positive integers k, r, we have

$$\mathcal{F}(G, r\binom{k+1}{2}) \subset \mathcal{F}(G,k,r).$$

Proof. Let $F \in \mathcal{F}(G, r\binom{k+1}{2})$, and consider a k-coloring c of the vertices of F and an r-coloring C of the edges of F. Since both vertices of each edge of F are colored, there are $\binom{k+1}{2}$ possible pairs of colors for the vertices of each edge. Therefore, the k-coloring c of the vertices of F and the r-coloring C of the edges of F determines a new $r\binom{k+1}{2}$-coloring C' of the edges of F, where the new color of each edge is given by its color in C and the colors of its endpoints in c. Since $F \in \mathcal{F}(G, r\binom{k+1}{2})$, there is an induced monochromatic copy of G in this new coloring C' of the edges of F. So, all edges in the original r-coloring of this copy of G must have been the same color, and all edges must have had the same pair of colors for its endpoints. Because G is non-bipartite, there does not exist a 2-coloring of the vertices of G such that the two vertices of each edge are different colors. Therefore, all vertices in this copy of G are the same color. Hence, every k-coloring of the vertices of F and r-coloring of the edges of F contains an induced copy of G that has all its vertices the same color and all its edges the same color. ∎

In the proof of Theorem 1, Nešetřil and Rödl showed that if G is bipartite, then there exists a bipartite graph $F \in \mathcal{F}(G,r)$. If $F = (V,E)$ is bipartite with bipartition $V = V_1 \cup V_2$, and if we color every vertex in V_1 red and every vertex in V_2 blue, then no edge has both of its vertices the same color. Hence, the non-bipartiteness assumption in Theorem 2 is necessary.

Corollary 1 follows from Theorem 1.

Corollary 1. For every graph G and for all positive integers k and r, $\mathcal{F}(G,k,r)$ is nonempty.

Proof. If G is an empty graph on n vertices, then by the pigeonhole principle, the empty graph on $k(n-1)+1$ vertices is our desired F. If G has an edge, then let l denote the number of isolated vertices of G and let G_1 denote the graph formed by removing the isolated vertices from G. Let the graph H be the disjoint union of G_1, l disjoint edges, and a C_5. Note that H is not bipartite and has no isolated vertices, $\mathcal{F}(H,k,r) \subset \mathcal{F}(G,k,r)$, and $\omega(H) = \omega(G)$. It follows from Theorem 2 that $\mathcal{F}(G,k,r)$ is nonempty. ∎

3. THE MINIMUM DEGREE OF A G-MINIMAL GRAPH

In this section we study the function $s(G)$ defined in the introduction. The *neighborhood* $N_H(v)$ of a vertex v in a graph H is the set of vertices adjacent to v.

Theorem 3. For all graphs G, we have

$$2\delta(G) - 1 \leq s(G) \leq R(G) - 1.$$

Proof. Assume for contradiction that there exists a G-minimal graph H with a vertex v of degree less than $2\delta(G) - 1$. Partition the neighborhood $N_H(v) = R \cup B$ such that $|R| \leq \delta(G) - 1$ and $|B| \leq \delta(G) - 1$. Color the edge (v, w) red if $w \in R$ and blue if $w \in B$. No matter how the remaining edges of H are colored, v is never a vertex of a monochromatic copy of G, since v has degree less than $\delta(G)$ in each color. Thus, H is not G-minimal, a contradiction. Therefore, $s(G) \geq 2\delta(G) - 1$.

For the upper bound on $s(G)$, we know by definition that $K_{R(G)} \rightarrow G$, so $K_{R(G)}$ has a G-minimal subgraph. Since every vertex of $K_{R(G)}$ has degree $R(G) - 1$, the minimum degree of a vertex in a subgraph of $K_{R(G)}$ is at most $R(G)-1$. Hence, $s(G) \leq R(G)-1$. ∎

We prove in Subsection 3.2. that the lower bound $s(G) \geq 2\delta(G) - 1$ is tight when G is a complete bipartite graph.

3.1. Complete Graphs

In this subsection, we prove that $s(K_k) = (k-1)^2$. We say a 2-coloring c of the edges of a graph T satisfies *Property* k if the following two conditions are satisfied:
(1) c does not contain a monochromatic copy of K_k.
(2) Let $T' = K_1 \otimes T$. Every 2-coloring of the edges of T' with the subgraph T maintaining the same coloring c contains a monochromatic copy of K_k.
Let $t(k)$ denote the smallest integer t such that there exists a graph T on t vertices with a 2-coloring of its edges that satisfies Property k. If v is a vertex of a K_k-minimal graph H, then the graph induced by $N_H(v)$ has a coloring that satisfies Property k. We therefore have the following lemma.

Lemma 1. For all positive integers k, we have

$$s(K_k) \geq t(k).$$

In fact, we will prove the following stronger result.

Theorem 4. For all positive integers k, we have

$$s(K_k) = t(k) = (k-1)^2.$$

Throughout the rest of the paper, we use $\bar{c}$ to denote the edge-coloring of $K_{(k-1)^2}$ consisting of $k-1$ disjoint blue copies of K_{k-1} and all the other edges colored red. In the following lemma, we find $t(k)$.

Lemma 2. For all positive integers k, we have

$$t(k) = (k-1)^2.$$

Proof. We first prove that the lower bound $t(k) \geq (k-1)^2$ holds for all positive integers k. Let T be a graph on $t < (k-1)^2$ vertices. Suppose we are given a fixed coloring c of the edges of T with the colors red and blue, without any monochromatic K_k. Let $T' = K_1 \otimes T$, and let v be the vertex added to T to obtain T'. We now construct a coloring of the edges of T' with the subgraph T maintaining the same coloring c and containing no monochromatic K_k.

If there are no blue K_{k-1}'s in the coloring c of T, then color every edge from v to a vertex in T blue. This coloring shows that c does not have Property k. Otherwise, choose the vertices of a monochromatic blue K_{k-1} from the t vertices of T. Pick out $k-1$ vertices (if possible) from the remaining $t-(k-1)$ vertices such that the graph induced by those $k-1$ vertices is a monochromatic blue K_{k-1}. Continue picking out monochromatic blue copies of K_{k-1} until there are no more monochromatic blue copies of K_{k-1} among the remaining vertices. Let j be the number of remaining vertices, and let b be the number of blue copies of K_{k-1} that were picked out. So, we have $t = b(k-1)+j$.

For every vertex in T that was picked out in one of the b disjoint monochromatic blue copies of K_{k-1}, we color the edge from that vertex to v red. For the remaining j vertices, we color the edges from those vertices to v blue. Since the j remaining vertices do not contain a monochromatic blue K_{k-1}, v is not a vertex of a monochromatic blue K_k. Since $j \geq 0$ and $t < (k-1)^2$, then $b < k-1$. Assume for contradiction that there are $k-1$ vertices of the b disjoint blue copies of K_{k-1} that form a monochromatic red K_{k-1}. By the Pigeonhole Principle, at least two of these $k-1$ vertices lie in the same monochromatic blue K_{k-1}, and so the edge between them must be blue, a contradiction. Therefore, v is not in a monochromatic red K_k. Hence, c does not have Property k, and since c was arbitrary, we have $t(k) \geq (k-1)^2$.

We now show that $t(k) \leq (k-1)^2$ holds for all positive integers k by showing that the coloring $\bar{c}$ of $K_{(k-1)^2}$ has Property k. It is clear that the coloring $\bar{c}$ does not have any monochromatic K_k, since the blue edges consist of disjoint copies of K_{k-1}, and the red subgraph is a complete $(k-1)$-partite graph. Therefore, the coloring $\bar{c}$ satisfies condition (1) of Property k.

Adjoin a vertex v to the vertices of $K_{(k-1)^2}$ to form $K_{(k-1)^2+1}$ and consider a coloring $\bar{c}'$ in which the $K_{(k-1)^2}$ subgraph keeps the coloring $\bar{c}$. If the graph induced by the blue neighborhood of v contains a monochromatic blue K_{k-1}, then adjoining v to this monochromatic blue K_{k-1} forms a monochromatic blue K_k. Thus, if $\bar{c}'$ does not have a monochromatic blue K_k, then v is adjacent by a red edge to at least one vertex from each of the $k-1$ disjoint monochromatic blue K_{k-1}'s, and these $k-1$ vertices along with v are the vertices of a monochromatic red K_k. Therefore, the coloring $\bar{c}$ satisfies condition (2) of Property k. Since $\bar{c}$ has Property k, we have $t(k) \leq (k-1)^2$, which shows that the lower bound proved earlier is tight. ∎

We say two r-colorings $c_1, c_2 : E \to \{0, \ldots, r-1\}$ of the edges of a graph $H = (V, E)$ are *isomorphic* if there exist bijections $\phi : V \to V$ and $\tau : \{0, \ldots, r-1\} \to \{0, \ldots, r-1\}$

such that both of the following hold:
(1) $(v,w) \in E$ if and only if $(\phi(v),\phi(w)) \in E$.
(2) If $(v,w) \in E$ then $\tau(c_1(v,w)) = c_2(\phi(v),\phi(w))$.

Let c be a 2-coloring of the edges of a graph H. We write $F \to (G, H^c)$ if every 2-coloring of the edges of F contains a monochromatic G or a coloring of H isomorphic to c.

We say the graph F is *Ramsey for* (G, H^c) if $F \not\to G$ but $F \to (G, H^c)$. We say that the pair (G, H^c) is *Ramsey* if there exists F that is Ramsey for (G, H^c). Trivially, (G, H^c) is not Ramsey if the coloring c of H contains a monochromatic copy of G. This new notation will be useful in proving our main result, $s(K_k) = (k-1)^2$.

Theorem 5. If the graphs $H_i \in \mathcal{F}(K_{k-1}, 2^{(i-1)(k-1)}, 2)$ for $1 \le i \le k-1$, then the product graph $F = H_1 \otimes H_2 \otimes \cdots \otimes H_{k-1}$ is Ramsey for $(K_k, K^{\bar{c}}_{(k-1)^2})$.

Proof. We first prove that $F \not\to K_k$. If (v,w) is an edge of F, v is a vertex in H_i, and w is a vertex in H_j, then color (v,w) blue if $i=j$ and red otherwise. Since each H_i does not contain a K_k, we know that there is no monochromatic blue K_k in this coloring. Since the red subgraph in this coloring of F is a complete $(k-1)$-partite graph, we know that there is no red K_k in this coloring. Hence, $F \not\to K_k$.

Next, we prove that $F \to (K_k, K^{\bar{c}}_{(k-1)^2})$. Assume we have a coloring of the edges of F with the colors red and blue without a monochromatic K_k. Since $H_1 \in \mathcal{F}(K_{k-1}, 1, 2)$, there must exist a monochromatic K_{k-1} in H_1. We may assume without loss of generality that this monochromatic K_{k-1} is blue. Denote the vertices of this monochromatic blue K_{k-1} by $v_1^{(j)}$, where $1 \le j \le k-1$. Now, we color each vertex v_2 of H_2 one of 2^{k-1} colors determined by the colors of the $k-1$ edges from v_2 to $\{v_1^{(j)}\}_{j=1}^{k-1}$. If two vertices v_2 and v_2' have the same color, then the edges $(v_2, v_1^{(j)})$ and $(v_2', v_1^{(j)})$ are the same color for $1 \le j \le k-1$. Since $H_2 \in \mathcal{F}(K_{k-1}, 2^{k-1}, 2)$, there are $k-1$ vertices $v_2^{(j)}$ with $1 \le j \le k-1$ of the same color whose edges form a monochromatic K_{k-1}.

Assume for contradiction that the edges of this monochromatic K_{k-1} in H_2 with vertices $\{v_2^{(j)}\}_{j=1}^{k-1}$ are red. If one of the edges $(v_1^{(j_1)}, v_2^{(j_2)})$ is red, then all the edges $(v_1^{(j_1)}, v_2^{(j)})$ with $1 \le j \le k-1$ are red because of how the vertex coloring of H_2 is defined. In this case, $\{v_1^{(j_1)}\} \cup \{v_2^{(j)}\}_{j=1}^{k-1}$ are the vertices of a monochromatic red K_k, contradicting the assumption that the given coloring of F did not contain a monochromatic K_k. So, all the edges $(v_1^{(j')}, v_2^{(j)})$ such that $1 \le j' \le k-1$ and $1 \le j \le k-1$ are blue. In this case, $\{v_1^{(j)}\}_{j=1}^{k-1} \cup \{v_2^1\}$ are the vertices of a monochromatic blue K_k, contradicting the assumption that the given coloring of F did not contain a monochromatic K_k. Therefore, the edges of this monochromatic K_{k-1} with vertices $\{v_2^{(j)}\}_{j=1}^{k-1}$ must be blue. If any edge $(v_1^{(j_1)}, v_2^{(j_2)})$ is blue, then the edges $(v_1^{(j_1)}, v_2^{(j)})$ are all blue for $1 \le j \le k-1$. In this case, $\{v_1^{(j_1)}\} \cup \{v_2^{(j)}\}_{j=1}^{k-1}$ are the vertices of a monochromatic blue K_k, contradicting the assumption that the given coloring of F did not contain a monochromatic K_k. So all the edges $(v_1^{(j_1)}, v_2^{(j_2)})$ with $1 \le j_1 \le k-1$ and $1 \le j_2 \le k-1$ are red.

We will induct on i from 3 to $k-1$. Color each vertex v_i of H_i one of $2^{(i-1)(k-1)}$ colors determined by the color of the edges from v_i to $v_l^{(j)}$ with $1 \leq l \leq i-1$ and $1 \leq j \leq k-1$. If two vertices v_i and v_i' have the same color, then the edges $(v_i, v_l^{(j)})$ and $(v_i', v_l^{(j)})$ are the same color for $1 \leq j \leq k-1$ and $1 \leq l \leq i-1$. Since $H_i \in \mathcal{F}(K_{k-1}, 2^{(i-1)(k-1)}, 2)$, then there are $k-1$ vertices $v_i^{(j)}$ with $1 \leq j \leq k-1$ of the same color whose edges form a monochromatic K_{k-1}. By the same argument that proved that the monochromatic K_{k-1} with vertices $\{v_2^{(j)}\}_{j=1}^{k-1}$ had to be blue, we have that the monochromatic K_{k-1} with vertices $\{v_i^{(j)}\}_{j=1}^{k-1}$ has to be blue. Likewise, the edges $(v_l^{(j_1)}, v_i^{(j_2)})$ with $1 \leq j_1 \leq k-1$, $1 \leq j_2 \leq k-1$, and $1 \leq l \leq i-1$ must all be red.

So, for each i with $1 \leq i \leq k-1$, $\{v_i^{(j)}\}_{j=1}^{k-1}$ are the vertices of a monochromatic blue K_{k-1}. Moreover, if $i, l, j_1, j_2 \in \{1, \ldots, k-1\}$ and $i \neq l$, then the edge $(v_l^{(j_1)}, v_i^{(j_2)})$ is red. So, the elements of the set $V = \{v_i^{(j)} : i, j \in \{1, \ldots, k-1\}\}$ are the vertices of a complete graph on $(k-1)^2$ vertices with the coloring $\bar{c}$. Therefore, $F \rightarrow (K_k, K_{(k-1)^2}^{\bar{c}})$. ∎

Now we prove Theorem 4.

Proof of Theorem 4: By Corollary 1, there exist graphs $H_i \in \mathcal{F}(K_{k-1}, 2^{(i-1)(k-1)}, 2)$ for $1 \leq i \leq k-1$. Let F be the product graph $H_1 \otimes H_2 \otimes \cdots \otimes H_{k-1}$ and V be the vertex set of F. By Theorem 5, F is Ramsey for $(K_k, K_{(k-1)^2}^{\bar{c}})$. Let F_1 be the supergraph of F obtained from F by adjoining a vertex v_S with neighborhood $N_{F_1}(v_S) = S$ for each subset $S \subset V$ with $|S| = (k-1)^2$. Since F is Ramsey for $(K_k, K_{(k-1)^2}^{\bar{c}})$ and the edge-coloring $\bar{c}$ of $K_{(k-1)^2}$ has Property k, then $F_1 \rightarrow K_k$ and F_1 has a K_k-minimal subgraph F_2 that contains a vertex of F_1 not in F. Therefore, F_2 has minimum degree at most $(k-1)^2$, and $s(K_k) \leq (k-1)^2$. This upper bound on $s(K_k)$ matches the lower bound $s(K_k) \geq t(k) = (k-1)^2$ proved in Lemma 1 and Lemma 2. Hence, $s(K_k) = t(k) = (k-1)^2$. ∎

3.2. Bipartite Graphs

For each complete bipartite graph $K_{a,b}$, we find another complete bipartite graph $K_{m,n}$ that is $K_{a,b}$-minimal.

Theorem 6. Let a and b be positive integers such that $a \leq b$. Let $m = 2a-1$ and $n = 2(b-1)\binom{2a-1}{a} + 1$. Then $K_{m,n}$ is $K_{a,b}$-minimal.

Proof. We first show that $K_{m,n} \rightarrow K_{a,b}$. Let V_1 and V_2 denote the independent sets of vertices in $K_{m,n}$ that are disjoint and of size m and n, respectively. Consider a 2-coloring of the edges of $K_{m,n}$ with colors red and blue. For each vertex v of degree $2a-1$, either the number of blue edges adjacent to v is at least a or the number of red edges adjacent to v is at least a. Therefore, for every 2-coloring of the edges of $K_{m,n}$, and for each vertex $v \in V_2$, there is at least one subset $S(v) \subset V_1$ with $|S(v)| = a$ such that the edges from $S(v)$ to v are all the same color. There are $\binom{2a-1}{a}$ possible subsets $S(v)$. Since there are $2(b-1)\binom{2a-1}{a} + 1$ vertices in V_2, then by the Pigeonhole Principle there exists $v_1, \ldots, v_{2b-1}$ with $S(v_1) = \cdots = S(v_{2b-1}) := S$. Since there are $2b-1$ such vertices, at least b of these vertices have only red edges adjacent to the vertices in S or at least b of these vertices have only blue edges adjacent to the vertices in S. Then these b vertices along with the vertices in S are the vertices of an induced monochromatic $K_{a,b}$.

Now we show that $K_{m,n} - e \not\to K_{a,b}$. We first note that for every pair of edges e_1, e_2 of $K_{a,b}$, there exists an isomorphism of $K_{a,b}$ that maps e_1 to e_2. Thus, $K_{m,n} - e$ is well-defined without specifying e. We give a 2-coloring of $K_{m,n} - e$ without a monochromatic $K_{a,b}$. Let w denote the only vertex of $K_{m,n} - e$ that has degree $2a-2$. Color $a-1$ of the edges that are adjacent to w red and the other $a-1$ edges that are adjacent to w blue. So, w is not a vertex of a monochromatic $K_{a,b}$ since the degree of w in a monochromatic subgraph is at most $a-1$. For each subset $S \subset V_1$ with $|S| = a$, pick out $b-1$ vertices of V_2 to have red edges adjacent to the vertices of S and blue edges adjacent to the vertices of $V_1 - S$, and then pick out $b-1$ vertices of V_2 to have blue edges adjacent to the vertices of S and red edges adjacent to the vertices of $V_1 - S$. We have colored all the edges of $K_{m,n} - e$, and there are no monochromatic $K_{a,b}$ in this coloring. Therefore, $K_{m,n} - e \not\to K_{a,b}$. Hence, $K_{m,n}$ is $K_{a,b}$-minimal. ∎

As a corollary of the previous theorem, we have $s(K_{a,b}) \leq 2\min(a,b) - 1$. Since the minimum degree of $K_{a,b}$ is $\min(a,b)$, then the upper bound on $s(K_{a,b})$ matches the lower bound $s(K_{a,b}) \geq 2\min(a,b) - 1$ in Theorem 3.

Corollary 2. For all positive integers a and b, we have

$$s(K_{a,b}) = 2\min(a,b) - 1.$$

Every bipartite graph H on v vertices is the subgraph of $K_{a,v-a}$ for some positive integer a with $a \leq \frac{v}{2}$. If H is a subgraph of $K_{a,v-a}$, $n = 2(a-v-1)\binom{2a-1}{a} + 1$, and $m = 2a-1$, then $K_{m,n}$ has a H-minimal subgraph. We therefore arrive at the following corollary of Theorem 6.

Corollary 3. If H is a bipartite graph with v vertices then

$$s(H) \leq v - 1.$$

We proved in Corollary 2 that the lower bound $s(G) \geq 2\delta(G) - 1$ is tight for complete bipartite graphs. The following question asks for which graphs is the lower bound tight.

Question 1. For which graphs G does $s(G) = 2\delta(G) - 1$?

4. CONCLUSION

While this paper determines the exact values of $s(G)$ if G is complete or complete bipartite, the exact value of $s(G)$ is open in most cases.

We have also introduced a new question in Ramsey theory: Which pairs (G, H^c) are Ramsey?

A famous result of Erdős [3] is the probabilistic lower bound $R(K_n) > 2^{\frac{n}{2}}$ on the Ramsey number for the complete graph on n vertices. Theorem 7 follows from Erdős' probabilistic lower bound on Ramsey numbers for complete graphs and a theorem of Prömel and Rödl [17].

Theorem 7 (Prömel and Rödl, [17]). Let k_1 be a fixed positive constant such that $k_1 < \frac{1}{2}$. Then there exists a constant $k_2 > 0$ such that for all positive integers n, if H is a graph with at most $k_2 n$ vertices, c is a 2-coloring of the edges of H, and $m = \lfloor 2^{k_1 n} \rfloor$, then K_m is Ramsey for (K_n, H^c).

The Prömel-Rödl theorem gives evidence to support the following conjecture.

Conjecture 1. For every 2-coloring c of a graph H without a monochromatic K_k, the pair (K_k, H^c) is Ramsey.

We next introduce new Ramsey-type numbers. If (G,H^c) is Ramsey, then we define the *dichromatic Ramsey number* $b(G, H^c)$ to be the least v such that there exists a graph F with v vertices that is Ramsey for (G, H^c). If (G,H^c) is not Ramsey, we define $b(G, H^c) = \infty$. Define $S(G)$ to be the minimum positive integer v such that there exists a G-minimal graph F with exactly v vertices and minimum degree $\delta(F) = s(G)$. For all graphs G, we have $S(G) \geq R(G)$, since no G-minimal graph has less than $R(G)$ vertices. For complete bipartite graphs $K_{a,b}$ with $b \geq a$, the following theorem shows that both $S(K_{a,b})$ and $R(K_{a,b})$ have lower and upper bounds which are expressed as an exponential function in a multiplied by a linear factor in b.

Theorem 8. For all positive integers a and b with $a \leq b$, we have

$$2(b-1)\binom{2a-1}{a} + 2a \geq S(K_{a,b}) \geq R(K_{a,b}) > (2\pi\sqrt{ab})^{\frac{1}{a+b}}(\frac{a+b}{e^2})2^{\frac{ab-1}{a+b}}.$$

Proof. The upper bound is the number of vertices in $K_{2a-1,2(b-1)\binom{2a-1}{a}+1}$, which we showed was $K_{a,b}$-minimal. The lower bound is due to Fan Chung and Ron Graham [1]. ∎

Since every K_k-minimal graph has a proper subgraph which is Ramsey for $(K_k, K^{\bar{c}}_{(k-1)^2})$, then $S(K_k) \geq b(K_k, K^{\bar{c}}_{(k-1)^2}) + 1$, which makes the next conjecture seem plausible.

Conjecture 2. For all positive integers k, we have

$$S(K_k) = 2^{\Omega(k^2)}.$$

Conjecture 2 would imply that $S(K_k)$ grows considerably faster than the Ramsey number $R(K_k)$.

4.1. Multicolored Generalizations

In this section, we consider the natural generalization to r-colorings. Most of the results in this paper carry over to r colors, and the proofs are straightforward generalizations. We outline these multicolored results below.

We say that H is $(G;r)$-*minimal* if $H \rightarrow (G;r)$ but $H' \not\rightarrow (G;r)$ for every proper subgraph H' of H. Let $s(G;r)$ denote the least nonnegative integer s such that there exists a $(G;r)$-minimal graph H with a vertex of degree s. Several of the results we obtained on $s(G)$ for 2 colors generalize naturally to r colors. We omit the proofs of these multicolored generalizations, as they are essentially the same proofs as those used for 2 colors.

Theorem 9. For all graphs G, we have

$$r\delta(G) - r + 1 \leq s(G;r) \leq R(G;r) - 1.$$

The proof of Theorem 6 can be easily generalized to prove $s(K_{a,b};r) = r\min(a,b) - r + 1$ for all positive integers a, b, and r.

Theorem 10. Let a and b be positive integers such that $a \leq b$. Let $m = ra - r + 1$ and $n = r(b-1)\binom{ra-r+1}{a} + 1$. Then $K_{m,n}$ is $K_{a,b}$-minimal.

While we proved $s(K_k) = (k-1)^2$, it is still an open problem to determine $s(K_k;r)$ for $r > 2$.

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