Crossings, colorings, and cliques

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Abstract

Albertson conjectured that if graph G has chromatic number r, then the crossing number of G is at least that of the complete graph K_r. This conjecture in the case $r = 5$ is equivalent to the four color theorem. It was verified for $r = 6$ by Oporowski and Zhao. In this paper, we prove the conjecture for $7 \leq r \leq 12$ using results of Dirac; Gallai; and Kostochka and Stiebitz that give lower bounds on the number of edges in critical graphs, together with lower bounds by Pach *et al.* on the crossing number of graphs in terms of the number of edges and vertices.

1 Introduction

For more than a century, from Kempe through Appel and Haken and continuing to the present, the Four Color Problem [5, 31] has played a leading role in the development of graph theory. For background we recommend the classic book by Jensen and Toft [18].

There are three classic relaxations of planarity. The first is that of a graph embedded on an arbitrary surface. Here Heawood established an upper bound for the number of colors needed to color any embedded graph. About forty years ago Ringel and Youngs completed the work of showing that the Heawood bound is (with the exception of Klein's bottle) sharp. Shortly thereafter Appel and Haken proved the Four Color Theorem. One consequence of these results is that the maximum chromatic number of a graph embedded on any given surface is achieved by a complete graph. Indeed, with the exception of the plane and Klein's bottle, a complete graph is the only critical graph with maximum chromatic number that embeds on a given surface.

The second classic relaxation of planarity is thickness, the minimum number of planar subgraphs needed to partition the edges of the graph. It is well known that thickness 2 graphs are 12-colorable and that K_8 is the largest complete graph with thickness 2. Sulanke showed that the 9-chromatic join of K_6 and C_5 has thickness 2. Thirty years later Boutin, Gethner, and Sulanke [7] constructed infinitely many 9-chromatic critical graphs of thickness 2. We do know that if G has thickness t, then G is $6t$-colorable. When $t \geq 3$, we do not know whether complete graphs have the maximum chromatic number among all graphs of thickness t. We do know that if $t \geq 3$, then K_{6t-2} is the largest complete graph with thickness t [4].

The third classic relaxation of planarity is crossing number. The *crossing number* of a graph G, denoted by $\mathrm{cr}(G)$, is defined as the minimum number of crossings in a drawing of G. There are subtleties to this definition and we suggest Szekely [33] for a look at foundational issues related to the

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crossing number and a survey of recent results. A bibliography of papers about crossings can be found at [34]. Surprisingly, there are only two papers that relate crossing number with chromatic number [2, 28]. Since these papers are not well known, we briefly review some of their results to set the context for our work.

Perhaps the first question one might ask about the connections between the chromatic number and the crossing number is whether the chromatic number is bounded by a function of the crossing number. Albertson [2] conjectured that $\chi(G) = \mathrm{O}(\mathrm{cr}(G)^{1/4})$ and this was shown by Schaefer [32]. In Section 5, we give a short proof of this.

Although few exact values are known for the crossing number of complete graphs, the asymptotics of this problem are well-studied. Guy conjectured [16] that the crossing number of the complete graph is as follows.

Conjecture 1 (Guy)**.**

$$\mathrm{cr}(K_n) = \frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor. \tag{1}$$

He verified this conjecture for $n \leq 10$ and Pan and Richter [30] recently confirmed it for $n = 11, 12$. Let $f(n)$ denote the right hand side of equation (1). It is easy to show that $f(n)$ is an upper bound for $\mathrm{cr}(K_n)$, by considering a particular drawing of K_n where the vertices are equally spaced around two concentric circles.

Kleitman proved that $\lim_{n\to\infty} cr(K_n)/f(n) \geq 0.80$ [19]. Recently de Klerk *et al.* [20] strengthened this lower bound to 0.83. By refining the techniques in [20], de Klerk, Pasechnik, and Schrijver [21] further improved the lower bound to 0.8594. These lower bounds shows that Schaefer's result that $\chi(G) = \mathrm{O}(\mathrm{cr}(G)^{1/4})$ is best possible.

The next natural step would be to determine exact values of the maximum chromatic number for small numbers of crossings. An easy application of the Four Color Theorem shows that if $\mathrm{cr}(G) = 1$, then $\chi(G) \leq 5$. Oporowski and Zhao [28] showed that the conclusion also holds when $\mathrm{cr}(G) = 2$. They further showed that if $\mathrm{cr}(G) = 3$ and G does not contain a copy of K_6, then $\chi(G) \leq 5$; they conjectured that this conclusion remains true even if $\mathrm{cr}(G) \in \{4, 5\}$. Albertson, Heenehan, McDonough, and Wise [3] showed that if $\mathrm{cr}(G) \leq 6$, then $\chi(G) \leq 6$.

The relationship between pairs of crossings was first studied by Albertson [2]. Given a drawing of graph G, each crossing is uniquely determined by the *cluster* of four vertices that are the endpoints of the crossed edges. Two crossings are said to be *dependent* if the corresponding clusters have at least one vertex in common, and a set of crossings is said to be *independent* if no two are dependent. Albertson gave an elementary argument proving that if G is a graph that has a drawing in which all crossings are independent, then $\chi(G) \leq 6$. He also showed that if G has a drawing with three crossings that are independent, then G contains an independent set of vertices one from each cluster. Since deleting this independent set leaves a planar graph, $\chi(G) \leq 5$. He conjectured that if G has a drawing in which all crossings are independent, then $\chi(G) \leq 5$. Independently, Wenger [35] and Harmon [17] showed that any graph with four independent crossings has an independent set of vertices one from each cluster, but there exists a graph with five independent crossings that contains no independent set of vertices one from each cluster. Finally, Král and Stacho [25] proved the conjecture that if G has a drawing in which all crossings are independent, then $\chi(G) \leq 5$.

Our purpose in this paper is to investigate whether for $r \geq 5$ the complete graphs are the unique critical r-chromatic graphs with minimum crossing number. We look for results analogous to those on coloring embedded graphs. Our problem may be more difficult, since we have nothing like a Heawood Theorem to give a good bound on the chromatic number in terms of the crossing number. Of course we also have nothing like a Ringel-Youngs Theorem to give the crossing number of the complete graph. To illustrate our ignorance, we recall that $\mathrm{cr}(K_n)$ is known only for $n \le 12$ and that the results for $n = 11$ (and thus $n = 12$) are recent [30].

At an AMS special session in Chicago in October of 2007, Albertson conjectured the following.

Conjecture 2 (Albertson)**.** *If* $\chi(G) \ge r$*, then* $\mathrm{cr}(G) \ge \mathrm{cr}(K_r)$.

At that meeting Schaefer observed that if G contains a subdivision of K_r, then such a subdivision must have at least as many crossings as K_r [32]. A classic conjecture attributed to Hajós was that if G is r-chromatic, then G contains a subdivision of K_r. Dirac [10] verified the conjecture for $r \le 4$. In 1979, Catlin [9] noticed that the lexicographic product of C_5 and K_3 is an 8-chromatic counterexample to the Hajós Conjecture. He generalized this construction to give counterexamples to Hajós' conjecture for all $r \ge 7$. A couple of years later Erdős and Fajtlowicz [13] proved that almost all graphs are counterexamples to Hajós' conjecture. However, Hajós' conjecture remains open for $r = 5, 6$. Note that if Hajós' conjecture does hold for a given G, then Alberton's conjecture also holds for that same G. This explains why Albertson's conjecture is sometimes referred to as the Weak Hajós Conjecture.

The rest of this paper is organized as follows. In Section 2 we discuss known lower bounds on the number of edges in r-critical graphs. In Section 3 we discuss known lower bounds on the crossing number, in terms of the number of edges. In Section 4 we prove Albertson's conjecture for $7 \le r \le 12$ by combining the results in the previous sections. In Section 5 we show that for $r \ge 37$, any minimal counterexample to this conjecture has less than $4r$ vertices, and we also give a few concluding remarks.

2 Color critical graphs

About 1950, Dirac introduced the concept of color criticality in order to simplify graph coloring theory, and it has since led to many beautiful theorems. A graph G is *r-critical* if $\chi(G) = r$ but all proper subgraphs of G have chromatic number less than r.

Let G denote an r-critical graph with n vertices and m edges. Define the excess $\epsilon_r(G)$ of G to be

$$\epsilon_r(G) = \sum_{x \in V(G)} (\deg(x) - (r-1)) = 2m - (r-1)n.$$

Since G is r-critical, every vertex has degree at least $r-1$ and so $\epsilon_r(G) \ge 0$. Brooks' theorem is equivalent to saying that equality holds if and only if G is complete or an odd cycle. Dirac [11] strengthened Brooks' theorem by proving that for $r \ge 3$, if G is not complete, then $\epsilon_r(G) \ge r-3$. Later, Dirac [12] gave a complete characterization for $r \ge 4$ of those r-critical graphs with excess $r-3$, and, in particular, they all have $2r-1$ vertices. Gallai [15] proved that r-critical graphs that are not complete and that have at most $2r-2$ vertices have much larger excess. Namely, if G has $n = r+p$ vertices and $2 \le p \le r-2$, then $\epsilon_r(G) \ge pr - p^2 - 2$. A fundamental difference between Gallai's bound and Dirac's bound is that Gallai's bound grows with the number of vertices (while Dirac's does not). Several other papers [14, 26, 24, 22] prove such Gallai-type bounds. Gallai further proved that r-critical graphs with at most $2r-2$ vertices are decomposable, i.e., their complement is disconnected. Kostochka and Stiebitz [23] proved that if $n \ge r+2$ and $n \ne 2r-1$, then $\epsilon_r(G) \ge 2r-6$.

We will frequently use the bounds due to Dirac and to Kostochka and Stiebitz. When we use these bounds, it will be convenient to rewrite them in terms of m, as below.

If G is r-critical and not a complete graph and $r \geq 3$, then

$$m \geq \frac{r-1}{2}n + \frac{r-3}{2}.$$

We call this Dirac's bound.

If G is r-critical, $n \geq r+2$, and $n \neq 2r-1$, then

$$m \geq \frac{r-1}{2}n + r - 3.$$

We call this the bound of Kostochka and Stiebitz.

We finish the section with a simple lemma classifying the r-critical graphs with at most $r+2$ vertices.

Lemma 1. *For $r \geq 3$, the only r-critical graphs with at most $r+2$ vertices are K_r and $K_{r+2} \setminus C_5$, the graph obtained from K_{r+2} by deleting the edges of a cycle of length five.*

Proof. The proof is by induction on r. For the base case $r = 3$, the 3-critical graphs are precisely odd cycles, and those with at most five vertices are K_3 and $C_5 = K_5 \setminus C_5$.

Let G be an r-critical graph with $r \geq 4$ and $n \leq r+2$ vertices, so all vertices of G have degree at least $r-1 \geq n-3$. If G has a vertex v adjacent to all other vertices of G, then clearly $G \setminus v$ is $(r-1)$-critical with at most $r+1$ vertices, and by induction, we are done in this case. So we may suppose every vertex in the complement of G has degree at least one and at most two. Since the number n of vertices of G is at most $r+2 \leq 2r-2$, Gallai's decomposition result implies that the vertex set of G can be partitioned $V(G) = V_1 \cup V_2$ such that all vertices in V_1 are adjacent to all vertices in V_2. Also, there are no two vertices u, w of G that have the same neighborhood, otherwise we could $(r-1)$-color $G \setminus u$ and give w the same color as u. This implies, for $i = 1, 2$, that the complement of the subgraph of G induced by V_i must contain a triangle or a path with three edges, and hence the subgraph of G induced by V_i has chromatic number at most $|V_i| - 2$. However, this implies $\chi(G) \leq |V_1| - 2 + |V_2| - 2 = n - 4 \leq r - 2$, contradicting the hypothesis that G is r-critical. $\square$

We remark that the same proof can be used to show for $r \geq 4$ that the only r-critical graph with $r+3$ vertices is $K_{r+3} \setminus C_7$.

3 Lower bounds on crossing number

A simple consequence of Euler's polyhedral formula is that every planar graph with $n \geq 3$ vertices has at most $3n - 6$ edges. Suppose G is a graph with n vertices and m edges. By deleting one crossing edge at a time from a drawing of G until no crossing edges exist, we see that

$$\mathrm{cr}(G) \geq m - (3n - 6). \tag{2}$$

Pach, R. Radoičić, G. Tardos, and G. Tóth [29] proved the following lower bounds on the crossing number.

$$\begin{aligned} \mathrm{cr}(G) &\geq \frac{7}{3}m - \frac{25}{3}(n-2), && (3)\\ \mathrm{cr}(G) &\geq 3m - \frac{35}{3}(n-2), && (4)\\ \mathrm{cr}(G) &\geq 4m - \frac{103}{6}(n-2). && (5) \end{aligned}$$

Although inequality (4) is not written explicitly in [29], it follows from their proof of (5). Of the above four inequalities on the crossing number, inequality (2) is best when $m \leq 4(n-2)$, inequality (3) is best when $4(n-2) \leq m \leq 5(n-2)$, inequality (4) is best for $5(n-2) \leq m \leq 5.5(n-2)$, and inequality (5) is best when $m \geq 5.5(n-2)$.

A celebrated result of Ajtai, Chvátal, Newborn, and Szemerédi [1] and Leighton [27], known as *the Crossing Lemma*, states that the crossing number of every graph G with n vertices and $m \geq 4n$ edges satisfies

$$\mathrm{cr}(G) \geq \frac{1}{64}\frac{m^3}{n^2}.$$

The constant factor $\frac{1}{64}$ comes from using inequality (2). The best known constant factor is due to Pach *et al.* [29]. Using (5), they show for $m \geq \frac{103}{6}n$ that

$$\mathrm{cr}(G) \geq \frac{1}{31.1}\frac{m^3}{n^2}. \qquad (6)$$

4 Albertson's conjecture for $r \leq 12$

In this section we prove Albertson's conjecture (Conjecture 2) for $r = 7, 8, 9, 10, 11, 12$. Note that if H is a subgraph of G, then $\mathrm{cr}(H) \leq \mathrm{cr}(G)$. Therefore, to prove Albertson's conjecture for a given r, it suffices to prove it only for r-critical graphs.

Lemma 1 demonstrates that the only r-critical graphs with $n \leq r+2$ vertices are K_r and $K_{r+2} \setminus C_5$. This second graph contains a subdivision of K_r. Indeed, by taking all the vertices of $K_{r+2} \setminus C_5$ and picking two adjacent vertices of degree $r-1$ to be internal vertices of a subdivided edge, we get a subdivision of K_r with only one subdivided edge. Hence, $\mathrm{cr}(K_{r+2} \setminus C_5) \geq \mathrm{cr}(K_r)$. So a counterexample to Albertson's conjecture must have at least $r+3$ vertices. However, none of our proofs rely on this observation except for the proof of Proposition 6; the others use only the easier observation that no r-critical graph has $r+1$ vertices.

Proposition 1. *If $\chi(G) = 7$, then $\mathrm{cr}(G) \geq 9 = \mathrm{cr}(K_7)$.*

Proof. By the remarks above, we may suppose G is 7-critical and not K_7. Let n be the number of vertices of G and m be the number of edges of G. By Dirac's bound, we have $m \geq 3n+2$. Borodin [6] showed that if a graph has a drawing in the plane in which each edge intersects at most one other edge, then the graph has chromatic number at most 6. Consider a drawing D of G in the plane with $\mathrm{cr}(G)$ crossings. Since G has chromatic number 7, there is an edge e in D that intersects at least two other edges. Beginning with e, we delete one crossing edge at a time, until no crossing edges exist. We get that $\mathrm{cr}(G) \geq m - (3n-6) + 1 = m - 3n + 7$. Since $m \geq 3n+2$, this bound gives:

$$\mathrm{cr}(G) \geq m - 3n + 7 \geq 9.$$

This completes the proof. ∎

Proposition 2. *If* $\chi(G) = 8$ *and* G *does not contain* K_8*, then* $\mathrm{cr}(G) \geq 20 > 18 = \mathrm{cr}(K_8)$.

Proof. We may suppose G is 8-critical. Let n be the number of vertices of G and m be the number of edges of G. When $n = 15$, Dirac's bound gives $m \geq \frac{7}{2}n + 2.5 = 55$, and thus inequality (3) gives

$$\mathrm{cr}(G) \geq \frac{7}{3}m - \frac{25}{3}(n-2) = 20.$$

When $n \neq 15$, the bound of Kostochka and Stiebitz gives $m \geq \frac{7}{2}n + 5$. When we substitute for m, inequalties (3) and (4) give

$$\mathrm{cr}(G) \geq m - 3n + 6 \geq \frac{n}{2} + 11,$$

and

$$\mathrm{cr}(G) \geq \frac{7}{3}m - \frac{25}{3}(n-2) \geq \frac{7}{3}(\frac{7}{2}n + 5) - \frac{25}{3}(n-2) = -\frac{n}{6} + \frac{85}{3}.$$

The first lower bound shows that $\mathrm{cr}(G) \geq 20$ if $n \geq 18$, while the second lower bound shows that $\mathrm{cr}(G) \geq 20$ if $n \leq 50$. This completes the proof. □

Proposition 3. *If* $\chi(G) = 9$ *and* G *does not contain* K_9*, then* $\mathrm{cr}(G) \geq 41 > 36 = \mathrm{cr}(K_9)$.

Proof. We may suppose G is 9-critical. Let $n \geq 11$ be the number of vertices of G and m be the number of edges of G. When $n = 17$, Dirac's bound gives $m \geq 4n + 3 = 71$, so inequality (3) gives

$$\mathrm{cr}(G) \geq \frac{7}{3}m - \frac{25}{3}(n-2) = \frac{122}{3} > 40.$$

Thus $\mathrm{cr}(G) \geq 41$. When $n \neq 17$, the bound of Kostochka and Stiebitz gives $m \geq 4n + 6$. Hence, inequality (3) gives

$$\mathrm{cr}(G) \geq \frac{7}{3}m - \frac{25}{3}(n-2) \geq n + \frac{92}{3} \geq 11 + \frac{92}{3} > 41.$$

This completes the proof. □

Proposition 4. *If* $\chi(G) = 10$ *and* G *does not contain* K_{10}*, then* $\mathrm{cr}(G) \geq 69 > 60 = \mathrm{cr}(K_{10})$.

Proof. We may suppose G is 10-critical. Let $n \geq 12$ be the number of vertices of G and m be the number of edges of G. When $n = 19$, Dirac's bound gives $m \geq \frac{9}{2}n + \frac{7}{2} = 89$, so inequality (4) gives

$$\mathrm{cr}(G) \geq 3m - \frac{35}{3}(n-2) = \frac{206}{3} > 68.$$

Thus $\mathrm{cr}(G) \geq 69$. When $n \neq 19$, the bound of Kostochka and Stiebitz gives $m \geq \frac{9}{2}n + 7$, so inequality (5) gives

$$\mathrm{cr}(G) \geq 4m - \frac{103}{6}(n-2) \geq \frac{5}{6}n + \frac{187}{3} \geq 10 + \frac{187}{3} > 72.$$

This completes the proof. □

Proposition 5. *If* $\chi(G) = 11$ *and* G *does not contain* K_{11}*, then* $\mathrm{cr}(G) \geq 104 > 100 = \mathrm{cr}(K_{11})$.

Proof. We may suppose G is 11-critical. Let $n \geq 13$ be the number of vertices of G and m be the number of edges of G. When $n = 21$, Dirac's bound gives $m \geq 5n + 4 = 109$, so inequality (5) gives

$$\operatorname{cr}(G) \geq 4m - \frac{103}{6}(n-2) = \frac{659}{6} > 109.$$

Thus $\operatorname{cr}(G) \geq 110$. When $n \neq 21$, the bound of Kostochka and Stiebitz gives $m \geq 5n+8$, so inequality (5) gives

$$\operatorname{cr}(G) \geq 4m - \frac{103}{6}(n-2) \geq \frac{17}{6}n + \frac{199}{3} \geq \frac{17}{6} \cdot 13 + \frac{199}{3} > 103.$$

Thus $\operatorname{cr}(G) \geq 104$, which completes the proof. □

Proposition 6. *If $\chi(G) = 12$, then $\operatorname{cr}(G) \geq 153 > 150 = \operatorname{cr}(K_{12})$.*

Proof. We may suppose G is 12-critical and is not K_{12}. Let n be the number of vertices of G and m be the number of edges of G. By the remark before the proof of Proposition 1, we may suppose G has at least 15 vertices.

Case 1: $n = 23$. Dirac's bound gives $m \geq \frac{11}{2}n + \frac{9}{2} = 131$, so inequality (5) gives

$$\operatorname{cr}(G) \geq 4m - \frac{103}{6}(n-2) = \frac{327}{2} > 163.$$

Thus $\operatorname{cr}(G) \geq 164$.

Case 2: $n > 16$ and $n \neq 23$. The bound of Kostochka and Stiebitz gives $m \geq \frac{11}{2}n+9$, so inequality (5) gives

$$\operatorname{cr}(G) \geq 4m - \frac{103}{6}(n-2) \geq \frac{29}{6}n + \frac{211}{3} > 152.$$

Thus we get $\operatorname{cr}(G) \geq 153$ if $n > 16$.

Case 3: $n = 15$. By rewriting Gallai's bound (from Section 2) as a lower bound on m, and substituting $r = 12$, we get the inequality $m \geq \frac{11}{2}n + \frac{3}{2}r - \frac{11}{2} = \frac{11}{2}n + \frac{25}{2} = 95$. Now inequality (5) gives

$$\operatorname{cr}(G) \geq 4m - \frac{103}{6}(n-2) > 4 \cdot 95 - \frac{103}{6} \cdot 13 > 156.$$

Case 4: $n = 16$. We again use Gallai's bound with $r = 12$, and now we get the inequality $m \geq \frac{11}{2}n + 2r - 9 = 103$. Now inequality (5) gives

$$\operatorname{cr}(G) \geq 4m - \frac{103}{6}(n-2) > 4 \cdot 103 - \frac{103}{6} \cdot 14 > 171.$$

This completes the proof. □

5 Concluding remarks

In the previous section, we showed that a minimal counterexample to Albertson's conjecture has at least $r + 3$ vertices. Here we give an upper bound on the number of vertices of a counterexample.

Proposition 7. *Suppose $r \geq 37$ and G is an r-critical graph with $n \geq 4r$ vertices and m edges. Then $\operatorname{cr}(G) \geq \operatorname{cr}(K_r)$.*

Proof. Since G is r-critical, $m \geq n(r-1)/2$. By assumption, $r \geq 37$ and $m \geq 18n > \frac{103}{6}n$. Therefore, the bound (6) gives

$$\mathrm{cr}(G) \geq \frac{1}{31.1}\frac{m^3}{n^2} \geq \frac{1}{8 \cdot 31.1}(r-1)^3 n \geq \frac{1}{64}(r-1)^3 r \geq \frac{1}{4}\lfloor\frac{r}{2}\rfloor\lfloor\frac{r-1}{2}\rfloor\lfloor\frac{r-2}{2}\rfloor\lfloor\frac{r-3}{2}\rfloor \geq \mathrm{cr}(K_r).$$

□

By modifying the above argument to assume only $n \geq r$, we can prove that $\mathrm{cr}(G) \geq (r-1)^4/2^8$ if G has chromatic number $r \geq 37$. This immediately implies $\chi(G) \leq 1 + 4\mathrm{cr}(G)^{1/4}$.

We think that if G has chromatic number r and does not contain K_r, then $\mathrm{cr}(G) - \mathrm{cr}(K_r)$ is not only nonnegative, but is at least cubic in r. Recall that $K_{r+2} \setminus C_5$ is r-critical and note that it is a subgraph of K_{r+2}; hence, if Guy's conjecture on the crossing number of K_r is true, then $K_{r+2} \setminus C_5$ shows that $\mathrm{cr}(G) - \mathrm{cr}(K_r)$ can be as small as cubic in r.

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