Complete minors and independence number

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Abstract

Let G be a graph with n vertices and independence number α. Hadwiger's conjecture implies that G contains a clique minor of order at least n/α. In 1982, Duchet and Meyniel proved that this bound holds within a factor 2. Our main result gives the first improvement on their bound by an absolute constant factor. We show that G contains a clique minor of order larger than $.504n/\alpha$. We also prove related results giving lower bounds on the order of the largest clique minor.

1 Introduction

A famous conjecture of Hadwiger [11] asserts that every graph with chromatic number k has a clique minor of order k. Hadwiger proved his conjecture for $k \leq 4$. Wagner [24] proved that the case $k = 5$ is equivalent to the Four Color Theorem. In a tour de force, Robertson, Seymour, and Thomas [20] settled the case $k = 6$ also using the Four Color Theorem. It is still open for $k \geq 7$.

Let G be a graph with n vertices, chromatic number k, and independence number α. In a proper k-coloring of G, each of the k independent sets has cardinality at most α, so $k \geq n/\alpha$. Hadwiger's conjecture therefore implies that every graph with n vertices and independence number α has a clique minor of order at least n/α. In 1982, Duchet and Meyniel [8] proved that this bound holds within a factor 2. More precisely, they showed that every graph with n vertices and independence number α contains a clique minor of order at least $\frac{n}{2\alpha-1}$.

There has been several improvements [16, 26, 18, 12, 13, 17, 2] on the bound of Duchet and Meyniel. Kawarabayashi, Plummer, and Toft [12] showed that every graph with n vertices and independence number $\alpha \geq 3$ contains a clique minor of order at least $\frac{n}{2\alpha-3/2}$, which was later improved to $\frac{n}{2\alpha-2}$ by Kawarabayashi and Song [13]. Wood [25] improves on the bound of Kawarabayashi, Plummer, and Toft on the order of a largest clique minor by an additive constant. Very recently, Balogh, Lenz, and Wu [2] improved these bounds to $\frac{n}{2\alpha-O(\log\alpha)}$.

Our main result gives the first improvement of these bounds by an absolute constant factor.

Theorem 1. *Let* $c = \frac{29-\sqrt{813}}{28} > .017$. *Every graph with* n *vertices and independence number* α *contains a clique minor of order at least* $\frac{n}{(2-c)\alpha}$.

Organization: In the next section, we prove Theorem 1, pending the proof of Lemma 1, which is proved in Section 3. In Section 4, we give a slight improvement on the order of the largest clique minor in a graph of independence number 2. In Section 5, we give a simple proof of a lower bound on the order of the largest clique minor in a graph, which improves on previous bounds when the chromatic number is at least almost linear in the number of vertices.

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Research supported in part by an NSF Graduate Research Fellowship and a Princeton Centennial Fellowship.

2 Proof of Theorem 1

We begin with some terminology. Let $G = (V, E)$ be a graph. The independence number and chromatic number of G are denoted by $\alpha(G)$ and $\chi(G)$, respectively. The graph G is *decomposable* if there is a vertex partition $V = V_1 \cup \ldots \cup V_i$ into $i > 1$ nonempty parts such that sum of the independence numbers of the subgraphs induced by $V_1, ..., V_i$ is equal to the independence number of G. Every non-decomposable graph is connected as the sum of the independence numbers of the connected components of a graph is the independence number of the graph.

Let S be a vertex subset of G. Let $G[S]$ and $G \setminus S$ denote the induced subgraphs of G with vertex sets S and $V \setminus S$, respectively. The set S is *connected* if $G[S]$ is connected. The set S is *dominating* if every vertex in $V \setminus S$ has a neighbor in S. Let $\alpha(S)$ and $c(S)$ denote the independence number and number of connected components, respectively, of $G[S]$. Define the *potential* of S to be $\phi(S) := 2\alpha(S) - c(S) - |S|$. A useful property of the potential function is that it is additive on disjoint vertex subsets with no edges between them. This follows from the fact that the independence number, the number of components, and the number of vertices are all additive on disjoint vertex subsets with no edges between them. For example, the potential of an independent set is 0.

A *claw* is the complete bipartite graph $K_{1,3}$. A graph is *claw-free* if it does not contain a claw as an induced subgraph. A key ingredient in the proof of Theorem 1 is a result of Chudnovsky and Fradkin [5] that gives an approximate version of Hadwiger's conjecture for claw-free graphs. It states that every claw-free graph with chromatic number k has a clique minor of order at least $\frac{2}{3}k$. The proof of this result uses the structure theorem for claw-free graphs due to Chudnovsky and Seymour [6]. We could alternatively use a related recent result of Fradkin [9], which also uses the structure theorem for claw-free graphs. It states that every connected claw-free graphs on n vertices with independence number $\alpha \geq 3$ has a clique minor of order at least $\frac{n}{\alpha}$.

The proof of Theorem 1 is by induction on n. The base case $n = 1$ is trivial. The induction hypothesis is that every graph G' with n' vertices and independence number α' with $n' < n$ contains a clique minor of order at least $\frac{n'}{(2-c)\alpha'}$.

We may suppose G is non-decomposable. Indeed, otherwise by the pigeonhole principle one of the vertex subsets in the decomposition satisfies $|V_j| \geq \frac{\alpha(V_j)}{\alpha}n$, and the induction hypothesis implies that the subgraph of G induced by V_j has a clique minor of order at least

$$\frac{|V_j|}{(2-c)\alpha(V_j)} \geq \frac{\frac{\alpha(V_j)}{\alpha}n}{(2-c)\alpha(V_j)} = \frac{n}{(2-c)\alpha},$$

which would complete the proof.

We pick out induced claws from G one by one until the remaining set of vertices is claw-free. Let m denote the number of vertex-disjoint induced claws we take out of G. The remaining induced subgraph, which we denote by G', is claw-free and has $n' := n - 4m$ vertices. By the result of Chudnovsky and Fradkin [5] discussed at the beginning of this section, G' has a clique minor of order at least

$$\frac{2}{3}\chi(G') \geq \frac{2}{3}\frac{n'}{\alpha(G')} \geq \frac{2}{3}\frac{n'}{\alpha}.$$

If $n' \geq (1 - 14c)n$, then G' has a clique minor of order at least

$$\frac{2}{3}\frac{(1-14c)n}{\alpha} = \frac{n}{(2-c)\alpha},$$

in which case the proof is complete. So we may suppose $n - 4m = n' < (1 - 14c)n$, which implies $m > \frac{7}{2}cn$.

Let F denote the graph whose vertices are the m induced claws, where two claws are adjacent if there is an edge in G with one vertex in the first claw and the other vertex in the second claw. Any clique minor in F has a corresponding clique minor of the same order in G. If $\alpha(F) \leq \frac{7}{2}c\alpha$, then the induction hypothesis implies F and hence G would contain a clique minor of order at least $\frac{m}{(2-c)\frac{7}{2}c\alpha} > \frac{n}{(2-c)\alpha}$, in which case the proof is complete. So we may suppose that $\alpha(F) > \frac{7}{2}c\alpha$. This means that G has more than $\frac{7}{2}c\alpha$ vertex-disjoint induced claws with no edges between them. Let X be the union of the vertex sets of these claws. A claw has potential 1, and since the potential is additive on disjoint sets with no edges between them, $\phi(X) > \frac{7}{2}c\alpha$. By Lemma 1 below, there is a connected dominating set S with $|S| \leq 2\alpha - \frac{2}{7}\phi(X) - 1 < (2-c)\alpha$. Since S is a connected dominating set, G contains a clique minor of order one more than the order of the largest clique minor of $G \setminus S$. By the induction hypothesis, $G \setminus S$ has a clique minor of order at least $\frac{n-|S|}{(2-c)\alpha} > \frac{n}{(2-c)\alpha} - 1$, so G has a clique minor of order at least $\frac{n}{(2-c)\alpha}$, completing the proof. $\square$

3 Small connected dominating sets

The goal of this section is to prove the following lemma, which finishes the proof of Theorem 1.

Lemma 1. *If $G = (V, E)$ is a non-decomposable graph with independence number α, and $X \subset V$ with $\phi(X) > 0$, then G has a connected dominating set with at most $2\alpha - \frac{2}{7}\phi(X) - 1$ vertices.*

There are two basic operations we can apply to a vertex subset X which do not decrease its potential, which we call GROW and CONNECT.

GROW: If adjacent vertices x and y are respectively at distance 1 and 2 from X, then add x and y to X to obtain a new set X'.

If G is connected, the operation GROW can be applied to X if and only if X is not a dominating set. Indeed, all vertices have distance at most one from X if X is a dominating set. If X is not a dominating set, let v be a vertex not adjacent to any vertex in X, and let x_1, x_2 be the two closest vertices to X on a shortest path from v to X. With this operation, $|X'| = |X| + 2$, $\alpha(X') \geq \alpha(X) + 1$, and $c(X') \leq c(X)$, so $\phi(X') \geq \phi(X)$.

CONNECT: Add a vertex to X to obtain a new set X' with $c(X') < c(X)$.

With this operation, $|X'| = |X| + 1$, $c(X') \leq c(X) - 1$, and $\alpha(X') \geq \alpha(X)$, so $\phi(X') \geq \phi(X)$. It is also easy to see that, if each connected component of X has positive potential, then applying GROW or CONNECT, each connected component of the resulting set has positive potential.

Lemma 2. *If X is a vertex subset of a graph $G = (V, E)$ with $\alpha(X) = \alpha(G)$, then G is decomposable or X is connected or we can apply* CONNECT *to X.*

Proof. Let $X_1, \ldots, X_i$ denote the vertex sets of the connected components of $G[X]$. We may suppose $i \geq 2$ as otherwise X is connected. We have $\alpha(X) = \alpha(X_1) + \cdots + \alpha(X_i)$. Let U_j denote the set of vertices in $V \setminus X$ that have at least one neighbor in X_j. Since $\alpha(X) = \alpha(G)$, X is a dominating set, and hence $V \setminus X = U_1 \cup \ldots \cup U_i$. We may suppose no vertex $v \in V \setminus X$ has neighbors in different connected components of $G[X]$, as otherwise $c(X \cup \{v\}) < c(X)$ and we can apply CONNECT to X. Hence, $U_1, \ldots, U_i$ forms a partition of $V \setminus X$. Let $V_j = X_j \cup U_j$, so $V_1, \ldots, V_i$ forms a vertex partition of G. Since $\alpha(X) = \alpha(G)$ and no vertex in V_j has a neighbor in $X \setminus X_j$, we have $\alpha(V_j) = \alpha(X_j)$. Hence, $\alpha(V_1) + \cdots + \alpha(V_j) = \alpha(G)$ and G is decomposable, which completes the proof. □

By repeatedly applying the previous lemma, we obtain the following corollary.

Corollary 1. *If G is a non-decomposable graph and X is a vertex subset with $\alpha(X) = \alpha(G)$, then there is a connected dominating set X' which contains X with $\phi(X') \geq \phi(X)$.*

Call a vertex subset X *tough* if we cannot apply GROW or CONNECT to it. If G is connected, X is tough if and only if every vertex in $V \setminus X$ has a neighbor in exactly one connected component of $G[X]$. In particular, every tough vertex subset of a connected graph is dominating.

Lemma 3. *If G is connected and X is a vertex subset with $\phi(X) > 0$, then there is a tough vertex subset Y with $\phi(Y) \geq \phi(X)$ and the vertex set of each connected component of $G[Y]$ has positive potential.*

Proof. Let $X_1, \ldots, X_i$ denote the vertex sets of the connected components of $G[X]$. Let X' consist of the union of all X_j with $\phi(X_j) > 0$. We have $\phi(X') \geq \phi(X) > 0$ since ϕ is additive on disjoint vertex subsets with no edges between them. We apply GROW and CONNECT repeatedly starting with X' until we get a tough set Y. The set Y contains X' and satisfies $\phi(Y) \geq \phi(X') \geq \phi(X)$. Since the vertex set of each connected component of $G[X']$ has positive potential, and GROW and CONNECT keep this property on each application, the vertex set of each connected component of $G[Y]$ has positive potential. □

There is one more operation that we will sometimes use which decreases the potential of a set by at most 1 in each application.

WEAK CONNECT: Add at most two vertices to X to obtain a new set X' with $c(X') < c(X)$.

Lemma 4. *If $G = (V, E)$ is connected and X is a dominating set which is not connected, then we can apply* WEAK CONNECT *to X.*

Proof. Let $X_1, \ldots, X_i$ with $i = c(X) > 1$ be the vertex sets of the connected components of $G[X]$. Let U_j denote the set of vertices in $V \setminus X$ which are adjacent to at least one vertex in X_j. Since X is dominating, each vertex in $V \setminus X$ is in some U_j. Let $V_j = X_j \cup U_j$. If there is a vertex v in U_j and $U_{j'}$ with $j \neq j'$, then adding v to X decreases the number of components. So we may suppose that $V = V_1 \cup \ldots \cup V_i$ is a partition of the vertex set of G. If there is a vertex $v_1 \in V_j$ adjacent to a vertex $v_2 \in V_{j'}$ with $j \neq j'$, then adding v_1 and v_2 to X decreases the number of connected components. Otherwise, there are no edges between $V_1, \ldots, V_i$, contradicting G is connected. □

We have one more lemma before the main result of this section.

Lemma 5. *Suppose $G = (V, E)$ is non-decomposable and vertex subset X is tough and not connected. Let $X_1, \ldots, X_i$ denote the vertex sets of the connected components of $G[X]$. Let V_j be the set of vertices in X_j or adjacent to a vertex in X_j. Then every X_j with $\phi(X_j) = 1$ satisfies $\alpha(V_j) = \alpha(X_j)$ or there is a tough vertex subset Y with $\phi(Y) \geq \phi(X) - 1$ and $\alpha(Y) > \alpha(X)$.*

Proof. The set X is a dominating set since it is a tough vertex subset of the connected graph G. Hence, every vertex of G is in at least one V_j. Since we cannot apply CONNECT to X, every vertex of G is in precisely one V_j, i.e., $V = V_1 \cup \ldots \cup V_i$ is a vertex partition of G. Consequently, for each j, there are no edges between V_j and $X \setminus X_j$. We may suppose there is j such that $\phi(X_j) = 1$ and $\alpha(V_j) > \alpha(X_j)$ as otherwise we are done. Let I_j be a maximum independent set in V_j and $X' = (X \setminus X_j) \cup I_j$. Since there are no edges between V_j and $X \setminus X_j$, there are no edges between I_j and $X \setminus X_j$. Hence, $\alpha(X') = \alpha(X) - \alpha(X_j) + |I_j| > \alpha(X)$. Also, $\phi(X') = \phi(X \setminus X_j) + \phi(I_j) = \phi(X) - \phi(X_j) + \phi(I_j) = \phi(X) - 1$, since $\phi(I_j) = 0$ and the potential function is additive on disjoint vertex subsets with no edges between them. Therefore, X' satisfies the desired properties except possibly it is not tough. We repeatedly apply the operations GROW and CONNECT starting with X' until we cannot anymore, and the resulting tough set Y satisfies $\alpha(Y) \geq \alpha(X') > \alpha(X)$ and $\phi(Y) \geq \phi(X') = \phi(X) - 1$. □

We next prove the main result of this section, Lemma 1.

Proof of Lemma 1: The graph G is connected as it is non-decomposable. By Lemma 3, there is a tough vertex subset Y with $\phi(Y) \geq \phi(X)$ and the vertex set of each connected component of $G[Y]$ has positive potential. This implies that $\phi(Y) \geq c(Y)$ as ϕ is integer-valued and additive on disjoint sets of vertices with no edges between them. Also, Y is dominating as it is a tough vertex subset of the connected graph G.

First suppose that $\alpha(Y) \leq \alpha - \frac{\phi(X)}{7}$. By Lemma 4, we can repeatedly apply WEAK CONNECT starting with Y until we get a connected dominating set S. Each time we apply WEAK CONNECT, we add at most two vertices and the number of components decreases. Hence,

$$\begin{aligned} |S| &\leq |Y| + 2(c(Y)-1) = 2\alpha(Y) - c(Y) - \phi(Y) + 2(c(Y)-1) = 2\alpha(Y) - 2 + c(Y) - \phi(Y) \\ &\leq 2\alpha(Y) - 2 \leq 2\alpha - \frac{2}{7}\phi(X) - 2, \end{aligned}$$

and we are done. So we may suppose that $\alpha(Y) > \alpha - \frac{\phi(X)}{7}$.

We can repeatedly apply Lemma 5 starting with Y until the resulting tough set Z is connected or satisfies the following property. Letting $Z_1, \ldots, Z_h$ denote the vertex sets of the connected components of $G[Z]$ and V_j consist of Z_j together with those vertices adjacent to a vertex in Z_j, every Z_j with $\phi(Z_j) = 1$ satisfies $\alpha(V_j) = \alpha(Z_j)$. Note that $\phi(Z) > \phi(X) - \frac{\phi(X)}{7} = \frac{6}{7}\phi(X)$ as $\alpha(Y) > \alpha - \frac{\phi(X)}{7}$ and the independence number increases while the potential goes down by at most 1 in each application of Lemma 5.

If Z is connected, then $\frac{6}{7}\phi(X) < \phi(Z) = 2\alpha(Z) - 1 - |Z| \leq 2\alpha - 1 - |Z|$, and hence $|Z| > 2\alpha - \frac{6}{7}\phi(X) - 1$, in which case we are done as Z is a connected dominating set. So we may suppose every Z_j with $\phi(Z_j) = 1$ satisfies $\alpha(V_j) = \alpha(Z_j)$.

Let $A \subset \{1, \ldots, h\}$ consist of those j such that $\phi(Z_j) = 1$, and B consist of the those j such that $\phi(Z_j) \geq 2$. Let $Z_A = \bigcup_{j \in A} Z_j$ and $Z_B = \bigcup_{j \in B} Z_j$. We have $\frac{6}{7}\phi(X) < \phi(Z) \leq \phi(Z_A) + \phi(Z_B)$ since ϕ is additive on disjoint sets with no edges between them. We consider two cases.

Case 1: $\phi(Z_A) \geq \frac{2}{7}\phi(X)$. Let $V_A = \bigcup_{j \in A} V_j$. Since $Z_A \subset V_A$ and $\alpha(V_j) = \alpha(Z_j)$ for $j \in A$,

$$\alpha(Z_A) \leq \alpha(V_A) \leq \sum_{j \in A} \alpha(V_j) = \sum_{j \in A} \alpha(Z_j) = \alpha(Z_A),$$

and hence $\alpha(V_A) = \alpha(Z_A)$. Let $I \subset V \setminus V_A$ be an independent set of order $\alpha(V \setminus V_A)$ and $W = Z_A \cup I$. Since Z is tough, there are no edges between Z_A and $V \setminus V_A$, and hence no edges between Z_A and I. We have $\phi(W) = \phi(Z_A) + \phi(I) = \phi(Z_A)$ since $\phi(I) = 0$ and ϕ is additive on vertex subsets with no edges between them. Also, $\alpha(W) = \alpha(Z_A) + \alpha(I) = \alpha(Z_A) + \alpha(V \setminus V_A) = \alpha(V_A) + \alpha(V \setminus V_A) \geq \alpha$, where the last inequality is because independence number is subadditive on vertex subsets. It follows that

$\alpha(W) = \alpha$. Applying Corollary 1, there is a connected dominating set S with $\phi(S) \geq \phi(W) = \phi(Z_A)$. We have $|S| = 2\alpha(S) - c(S) - \phi(S) \leq 2\alpha - 1 - \phi(Z_A) \leq 2\alpha - 1 - \frac{2}{7}\phi(X)$, which completes this case.

Case 2: $\phi(Z_B) > \frac{4}{7}\phi(X)$. In this case, we apply GROW repeatedly starting with Z_B until we get a dominating set D which contains Z_B. The set D satisfies $\phi(D) \geq \phi(Z_B)$ and $c(D) \leq c(Z_B)$. By Lemma 4, we can then apply WEAK CONNECT until we get a connected dominating set S. Each application of WEAK CONNECT decreases the potential by at most one while decreasing the number of components by at least one. Hence,

$$\begin{aligned} \phi(S) &\geq \phi(D) - c(D) + 1 \geq \phi(Z_B) - c(Z_B) + 1 = 1 + \sum_{j \in B} (\phi(Z_j) - 1) \geq 1 + \sum_{j \in B} \frac{\phi(Z_j)}{2} \\ &= 1 + \frac{\phi(Z_B)}{2} > 1 + \frac{2}{7}\phi(X). \end{aligned}$$

We therefore have $|S| = 2\alpha(S) - \phi(S) - 1 < 2\alpha - \frac{2}{7}\phi(X) - 1$, which completes the proof. $\square$

4 Clique minors in graphs of independence number 2

Let G be a graph with n vertices, clique number ω, and independence number 2. The result of Duchet and Meyniel demonstrates that G has a clique minor of order $\frac{n}{3}$. Seymour [21] and independently Mader has asked whether the factor 1/3 can be improved. This question appears difficult and has received much attention recently. It was shown by Plummer, Stiebitz, and Toft [18] that G contains a clique minor of order at least $\frac{n+\omega}{3}$. Since Ajtai, Komlos, and Szemeredi [1] proved that $\omega \geq c\sqrt{n \log n}$ for some absolute constant $c > 0$, this gives a lower order improvement to the bound of Duchet and Meyniel. Kim [14] proved that this lower bound on the clique number of graphs of independence number 2 is tight apart from the constant factor. A recent result of Chudnovsky and Seymour [7] shows that if $\omega \geq \frac{n}{4}$ and n is even or $\omega \geq \frac{n+3}{4}$ and n is odd, then G contains a clique minor of order at least $\frac{n}{2}$. The lower bound on the order of the largest clique minor was later improved by Füredi, Gyárfás, and Simonyi [10] to $\frac{n}{3} + cn^{2/3}$. More recently, Blasiak [3] improved the bound further to $\frac{n}{3} + cn^{4/5}$. Here we give another improvement.

Theorem 2. *There is an absolute constant $c > 0$ such that every graph with n vertices and independence number 2 contains a clique minor of order at least $\frac{n}{3} + cn^{4/5}\log^{1/5} n$.*

The main purpose of this section is to prove Theorem 2. Let G be a graph with n vertices and independence number 2. A set of pairwise disjoint edges $e_1, \ldots, e_t$ of G is called a *connected matching of size t* if for every $1 \leq i < j \leq t$, there is an edge of G connecting a vertex in e_i to a vertex in e_j. Recall that the result of Duchet and Meyniel demonstrates that G has a clique minor of order at least $\frac{n}{3}$. Thomassé observed that improving the constant factor $\frac{1}{3}$ is equivalent to proving that G contains a connected matching of size at least cn, where $c > 0$ is an absolute constant. More precisely, in [12] it is shown that if G has a connected matching of size t, then G contains a clique minor of order at least $\frac{n+t}{3}$. In the other direction, if G has a clique minor of order $\frac{n}{3} + t$, then G contains a connected matching of size at least $\frac{3(t-1)}{4}$. Also using the fact that every clique of order ω contains a connected matching of size $\lfloor \frac{\omega}{2} \rfloor$, Theorem 2 follows from the next theorem.

Theorem 3. *For n sufficiently large, every graph $G = (V, E)$ with n vertices and independence number 2 has a clique or a connected matching of size at least $\frac{1}{6}n^{4/5}\log^{1/5} n$.*

We use the following well known bound ([4], Lemma 12.16) on the independence number of a graph with few triangles (see also [1] for a more general result).

Lemma 6. *Let G be a graph with n vertices, average degree at most d, and at most m triangles. Then G has an independent set of order*

$$\frac{4n}{39d}\left(\log d - \frac{1}{2}\log\frac{m}{n}\right).$$

Proof of Theorem 3: Let Δ denote the maximum degree of the complement $\overline{G}$ of G. Since G has independence number 2, the nonneighbors of a vertex form a clique in G. We may therefore suppose that $\Delta < \frac{1}{6}n^{4/5}\log^{1/5} n$. The number of edges of G is at least $\binom{n}{2} - \frac{n\Delta}{2} \geq \frac{n^2}{4}$.

For an edge e of G, let $N(e)$ denote the set of vertices not adjacent to either vertex of e. By counting over edges of G, the number of unordered triples of vertices containing exactly one edge is $\sum_{e\in E}|N(e)|$. Since each vertex of G has at most Δ nonneighbors, this number is at most $\binom{\Delta}{2}n \leq \frac{\Delta^2 n}{2}$. As G has at least $\frac{n^2}{4}$ edges, the average value of $|N(e)|$ is at most $\frac{\Delta^2 n/2}{n^2/4} = \frac{2\Delta^2}{n}$. Let $E' \subset E$ consist of those edges e with $|N(e)| \leq \frac{4\Delta^2}{n}$, so $|E'| \geq \frac{n^2}{8}$.

Let H be the graph with vertex set E such that $e_1, e_2 \in E$ are adjacent in H if and only if they do not share a vertex and there is a vertex in e_1 adjacent to a vertex in e_2 in G. Note that the vertices of a clique in H are the edges of a connected matching in G. Also, $e, e' \in E$ are not adjacent in H if and only if e' shares a vertex with e or both of the vertices of e' are in $N(e)$. Let $d = \frac{1}{100}n^{6/5}\log^{4/5} n$. Hence, the degree of $e \in E'$ in $\overline{H}$ is at most

$$2n + \binom{|N(e)|}{2} \leq 2n + \frac{1}{2}\left(\frac{4\Delta^2}{n}\right)^2 \leq 2n + 8\cdot 6^{-4}n^{6/5}\log^{4/5} n < d,$$

where we used the upper bound on Δ and that n is sufficiently large.

Let H' be the induced subgraph of $\overline{H}$ with vertex set E'. We will next give an upper bound on the number of triangles in H'. No three edges of G have disjoint vertices and form a triangle in $\overline{H}$ as otherwise a set of three vertices consisting of one vertex from each of these three edges would form an independent set in G of order 3, a contradiction. If three edges of G span exactly five vertices of G and are the vertices of a triangle in $\overline{H}$, then for one of the three edges, say e, we have that the other two edges share a vertex and all three of their vertices lie in $N(e)$. There are at most $\binom{n}{4}\binom{6}{3}$ triples of edges of G that span at most four vertices as we can first pick the vertices and then the edges. Using the upper bound on $|N(e)|$ for $e \in E'$, the number of triangles in H' is at most

$$\sum_{e\in E'} 3\binom{|N(e)|}{3} + \binom{n}{4}\binom{6}{3} \leq |E'|\frac{1}{2}\left(\frac{4\Delta^2}{n}\right)^3 + n^4 \leq \frac{n^2}{2}\frac{32\Delta^6}{n^3} + n^4 \leq 16n^{-1}\left(\frac{1}{6}n^{4/5}\log^{1/5} n\right)^6 + n^4 \leq 2n^4.$$

By Lemma 6, there is an independent set in H' of order at least

$$\frac{4|E'|}{39d}\left(\log d - \frac{1}{2}\log\frac{2n^4}{|E'|}\right) \geq \frac{n^2}{80d}\left(\log d - \frac{1}{2}\log(16n^2)\right) \geq \frac{n^2}{400d}\log n = \frac{1}{4}n^{4/5}\log^{1/5} n.$$

This independent set in H' is a clique in H and hence a connected matching in G. $\square$

5 Proof of Proposition 4

A well known result of Kostochka [15] and independently Thomason [22] demonstrates that every graph with average degree d contains a clique minor of order at least $\Omega\left(\frac{d}{\sqrt{\log d}}\right)$, and this is tight apart from the constant factor. Later, the exact constant factor was determined by Thomason [23]. This result implies that every graph with chromatic number k has a clique minor of order $\Omega\left(\frac{k}{\sqrt{\log k}}\right)$. We improve this bound for graphs of chromatic number almost linear in the number of vertices. We do not optimize the constant factors in the following proposition.

Proposition 4. *Every graph G with n vertices and chromatic number k contains a clique minor of order at least $\left(4\ln\frac{4n}{k}\right)^{-1}k$.*

We prove Proposition 4 by showing that any graph with large chromatic number has a large induced subgraph with small independence number. We may suppose $k \geq 4$ as otherwise the result is trivial. Let G be a graph with n vertices and chromatic number k. Suppose for contradiction that G does not contain a clique minor of order at least $\left(4\ln\frac{4n}{k}\right)^{-1}k$. By the result of Duchet and Meyniel [8], every induced subgraph of G with n' vertices has independence number greater than $2\frac{n'}{k}\ln\frac{4n}{k}$. Let $G_0 = G$. If G_i is already defined, let G_{i+1} be an induced subgraph of G obtained by deleting a maximum independent set from G_i. Let n_i denote the number of vertices of G_i, so $n_0 = n$. Note that $\chi(G_{i+1}) \geq \chi(G_i) - 1$ since G_{i+1} is obtained from G_i by deleting an independent set. It follows that $\chi(G_i) \geq k - i$. Also, $n_{i+1} < n_i - 2\frac{n_i}{k}\ln\frac{4n}{k} = \left(1 - \frac{2}{k}\ln\frac{4n}{k}\right)n_i$ as the independence number of G_i is greater than $2\frac{n_i}{k}\ln\frac{4n}{k}$. This inequality implies that

$$n_i < \left(1 - \frac{2}{k}\ln\frac{4n}{k}\right)^i n \leq e^{-2ik^{-1}\ln\frac{4n}{k}}n.$$

Letting $i = \lceil\frac{3k}{4}\rceil$, we have $n_i < \left(\frac{4n}{k}\right)^{-3/2}n \leq \frac{k}{8}$. However, $\chi(G_i) \geq k - i \geq \frac{k}{8}$ as $k \geq 4$, and so the number n_i of vertices of G_i is less than the chromatic number of G_i, a contradiction. □

Acknowledgments. I would like to thank Sasha Fradkin, Sasha Kostochka, Paul Seymour, Benny Sudakov, and the referee for many helpful comments.

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