On chromatic-$(5,4)$-colourings

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This is a companion note to the paper [1] in which we elucidate a comment made in the concluding remarks of that paper. We say that an edge coloring of K_n is a chromatic-(p,q)-coloring if every subgraph with chromatic number p receives at least q colors on its edges. Equivalently, an edge coloring is a chromatic-(p,q)-coloring if the union of every $q-1$ color classes has chromatic number at most $p-1$. In [1] and [2], we ask whether there is a chromatic-$(p,p-1)$-coloring of K_n using a subpolynomial number of colors. In [1], we showed that there is a chromatic-$(4,3)$-coloring of K_n with $2^{O(\sqrt{\log n})}$ colors. The purpose of this note is to show that the same coloring, originally found by Mubayi [3], is also a chromatic-$(5,4)$-coloring.

Let m and t be positive integers and let $n = m^t$. Identify the vertex set of K_n with $[m]^t$ and consider the edge-coloring function c_M defined over pairs of vertices $v = (v_1, \ldots, v_t)$ and $w = (w_1, \ldots, w_t)$ as follows:

$$c_M(v,w) = \Big(\{v_i, w_i\}, a_1, a_2, \ldots, a_t\Big),$$

where i is the minimum index j for which $v_j \neq w_j$ and $a_i = 1$ if $v_i \neq w_i$ and 0 otherwise. For t about $\sqrt{\log n}$ and $m = 2^t$, this gives a coloring of K_n with $2^{O(\sqrt{\log n})}$ colors.

We will prove that c_M is a chromatic-$(5,4)$-coloring. Let c_1, c_2 and c_3 be three colors used in the coloring c_M and let $\mathcal{G}$ be the graph induced by these three colors. It suffices to prove that $\mathcal{G}$ is 4-colorable. For each $j = 1,2,3$, let

$$c_j = \Big(\{x_j, y_j\}, a_{j,1}, a_{j,2}, \ldots, a_{j,t}\Big).$$

Furthermore, for each $j = 1,2,3$, let i_j be the minimum index i for which $a_{j,i} = 1$. Without loss of generality, we may assume that $i_1 \leq i_2 \leq i_3$. There are several different cases that we must consider, depending on the values of i_1, i_2, i_3 and a_{1,i_2}, a_{1,i_3}, and a_{2,i_3}. In the forthcoming figures, these cases will be represented by the following matrix:

$$\begin{pmatrix} a_{1,i_1} & a_{1,i_2} & a_{1,i_3} \\ a_{2,i_1} & a_{2,i_2} & a_{2,i_3} \\ a_{3,i_1} & a_{3,i_2} & a_{3,i_3} \end{pmatrix}.$$

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Case I. $i_1 < i_2 < i_3$

Note that $a_{2,i_1} = a_{3,i_1} = a_{3,i_2} = 0$. Define $\pi_1 : [m]^t \to \{0,1\}$ as

$$\pi_1(v) = \begin{cases} 0 & \text{if } v_{i_1} = x_1 \\ 1 & \text{if } v_{i_1} \neq x_1 \end{cases}$$

and $\pi_2, \pi_3 : [m]^t \to \{0,1,*\}$ as

$$\pi_2(v) = \begin{cases} 0 & \text{if } v_{i_2} = x_2 \\ 1 & \text{if } v_{i_2} = y_2 \\ * & \text{otherwise} \end{cases} \quad \text{and} \quad \pi_3(v) = \begin{cases} 0 & \text{if } v_{i_3} = x_3 \\ 1 & \text{if } v_{i_3} = y_3 \\ * & \text{otherwise} \end{cases}.$$

Define a map $\pi : [m]^t \to \{0,1\} \times \{0,1,*\} \times \{0,1,*\}$ as

$$\pi(v) = (\pi_1(v), \pi_2(v), \pi_3(v)),$$

and consider the graph $\pi(\mathcal{G})$. Since we can always pull back a proper vertex coloring of $\pi(\mathcal{G})$ into a proper vertex coloring of $\mathcal{G}$, it suffices to prove that $\pi(\mathcal{G})$ has chromatic number at most 4.

Note that since $a_{3,i_1} = a_{3,i_2} = 0$, any edge xy of color c_3 has $\pi_1(x) = \pi_1(y)$, $\pi_2(x) = \pi_2(y)$ and $\{\pi_3(x), \pi_3(y)\} = \{0,1\}$. We will use this fact in each of the subcases below.

Case I-A. $a_{2,i_3} = 0$.

Since $a_{2,i_1} = a_{2,i_3} = 0$, we see that any edge xy of color c_2 has $\pi_1(x) = \pi_1(y)$, $\pi_3(x) = \pi_3(y)$ and $\{\pi_2(x), \pi_2(y)\} = \{0,1\}$. Since all edges of color c_1 go between $\{0\} \times \{0,1,*\} \times \{0,1,*\}$ and $\{1\} \times \{0,1,*\} \times \{0,1,*\}$, it is now easy to see that the subgraphs of $\pi(\mathcal{G})$ induced on $\{0\} \times \{0,1,*\} \times \{0,1,*\}$ and $\{1\} \times \{0,1,*\} \times \{0,1,*\}$ are both bipartite. Hence $\pi(\mathcal{G})$ is 4-colorable.

Case I-B. $a_{2,i_3} = 1$.

In this case, any edge xy of color c_2 has $\pi_1(x) = \pi_1(y)$, $\{\pi_2(x), \pi_2(y)\} = \{0,1\}$ and either $\{\pi_3(x), \pi_3(y)\} = \{0,1\}$ or $* \in \{\pi_3(x), \pi_3(y)\}$. To analyze the edges of color c_1, we will split into some further subcases, noting that $\{\pi_1(x), \pi_1(y)\} = \{0,1\}$ in all cases.

Case I-B(i). $a_{1,i_2} = a_{1,i_3} = 0$.

Any edge xy of color c_1 has $\pi_2(x) = \pi_2(y)$ and $\pi_3(x) = \pi_3(y)$. Taking

$$\begin{aligned} V_1 &= \Big(\{0\} \times \{0,1,*\} \times \{0\}\Big) \cup \Big(\{1\} \times \{0\} \times \{1,*\}\Big), \\ V_2 &= \Big(\{0\} \times \{0\} \times \{1,*\}\Big) \cup \Big(\{1\} \times \{1,*\} \times \{1,*\}\Big), \\ V_3 &= \Big(\{0\} \times \{1,*\} \times \{1,*\}\Big) \cup \Big(\{1\} \times \{0,1,*\} \times \{0\}\Big), \end{aligned}$$

gives a proper coloring.

Case I-B(ii). $a_{1,i_2} = 0$, $a_{1,i_3} = 1$.

Any edge xy of color c_1 has $\pi_2(x) = \pi_2(y)$ and either $\{\pi_3(x), \pi_3(y)\} = \{0, 1\}$ or $* \in \{\pi_3(x), \pi_3(y)\}$. Taking

$$\begin{aligned}
V_1 &= \{0,1\} \times \{0,1,*\} \times \{0\}, \\
V_2 &= \Big(\{0\} \times \{0\} \times \{1,*\}\Big) \cup \Big(\{1\} \times \{1,*\} \times \{1,*\}\Big), \\
V_3 &= \Big(\{0\} \times \{1,*\} \times \{1,*\}\Big) \cup \Big(\{1\} \times \{0\} \times \{1,*\}\Big),
\end{aligned}$$

gives a proper coloring.

Case I-B(iii). $a_{1,i_2} = 1$, $a_{1,i_3} = 0$.

Any edge xy of color c_1 has $\pi_3(x) = \pi_3(y)$ and either $\{\pi_2(x), \pi_2(y)\} = \{0, 1\}$ or $* \in \{\pi_2(x), \pi_2(y)\}$. Taking

$$\begin{aligned}
V_1 &= \Big(\{0\} \times \{0,1,*\} \times \{0\}\Big) \cup \Big(\{1\} \times \{1,*\} \times \{1,*\}\Big), \\
V_2 &= \{0,1\} \times \{0\} \times \{1,*\}, \\
V_3 &= \Big(\{0\} \times \{1,*\} \times \{1,*\}\Big) \cup \Big(\{1\} \times \{0,1,*\} \times \{0\}\Big),
\end{aligned}$$

gives a proper coloring.

Case I-B(iv). $a_{1,i_2} = a_{1,i_3} = 1$. Any edge xy of color c_1 has either $\{\pi_2(x), \pi_2(y)\} = \{0, 1\}$ or $* \in \{\pi_2(x), \pi_2(y)\}$ and either $\{\pi_3(x), \pi_3(y)\} = \{0, 1\}$ or $* \in \{\pi_3(x), \pi_3(y)\}$. Taking

$$\begin{aligned}
V_1 &= \{0,1\} \times \{0,1,*\} \times \{0\}, \\
V_2 &= \{0\} \times \{1,*\} \times \{1,*\}, \\
V_3 &= \{1\} \times \{1,*\} \times \{1,*\}, \\
V_4 &= \{0,1\} \times \{0\} \times \{1,*\},
\end{aligned}$$

gives a proper coloring.

See Figure 1 for an illustration.

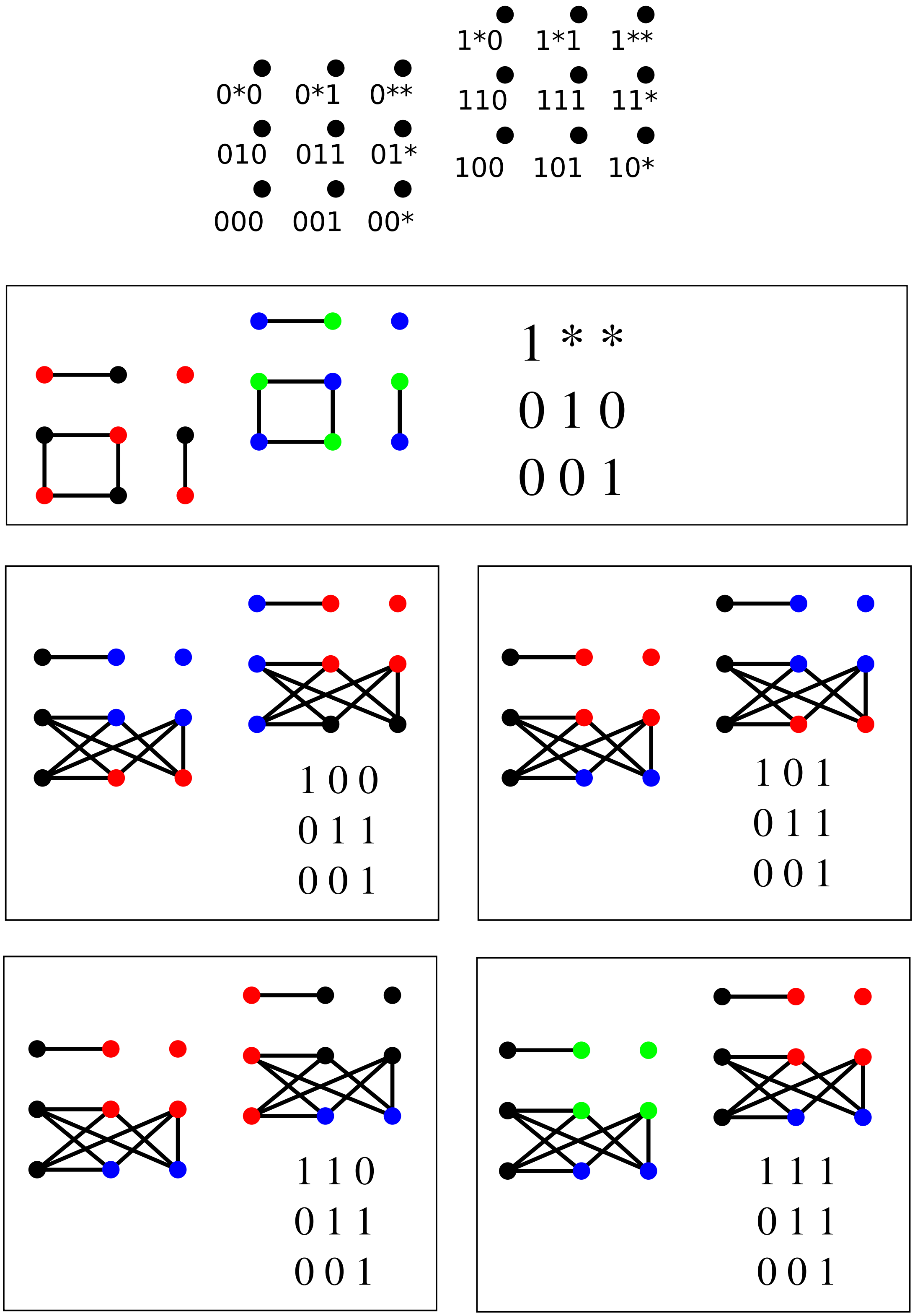

Figure 1: Colorings for Case I

Since the remaining cases are similar, we will give fewer explicit details.

Case II. $i_1 < i_2 = i_3$

For a vector $v \in [m]^t$, define

$$\pi_1(v) = \begin{cases} 0 & \text{if } v_{i_1} = x_1 \\ 1 & \text{if } v_{i_1} \neq x_1 \end{cases}.$$

Let $I = \{x_2, y_2\} \cap \{x_3, y_3\}$. Depending on whether $|I| = 0, 1, 2$, define $\pi_2(v)$ as

$$\pi_2(v) = \begin{cases} 0 & \text{if } v_{i_2} = x_2 \\ 1 & \text{if } v_{i_2} = y_2 \\ 2 & \text{if } v_{i_2} = x_3 \\ 3 & \text{if } v_{i_2} = y_3 \\ * & \text{otherwise} \end{cases} \quad \pi_2(v) = \begin{cases} 0 & \text{if } v_{i_2} = x_2 \\ 1 & \text{if } v_{i_2} = y_2 \\ 2 & \text{if } v_{i_2} = y_3 \\ * & \text{otherwise} \end{cases} \quad \pi_2(v) = \begin{cases} 0 & \text{if } v_{i_2} = x_2 \\ 1 & \text{if } v_{i_2} = y_2 \\ * & \text{otherwise} \end{cases},$$

respectively, where for the second case, we are assuming that $y_3 \notin \{x_2, y_2\}$. Define $\pi(v) = (\pi_1(v), \pi_2(v))$.

Case II-A. $a_{1,i_2} = 0$.

One can easily check that $\pi(\mathcal{G})$ is bipartite.

Case II-B. $a_{1,i_2} = 1$

If $I = \emptyset$, then a 4-coloring of $\pi(\mathcal{G})$ is given by

$$V_1 = \{(0,0), (0,2), (0,*)\},\ V_2 = \{(0,1), (0,3)\},$$
$$V_3 = \{(1,0), (1,2), (1,*)\},\ V_4 = \{(1,1), (1,3)\}.$$

If $|I| = 1$, then define $V_1 = \{(0,0), (0,2), (0,*)\}, V_2 = \{(1,0), (1,2), (1,*)\}$ and $V_3 = \{(0,1), (1,1)\}$. Finally, if $|I| = 2$, then define $V_1 = \{(0,0), (0,*)\}, V_2 = \{(1,0), (1,*)\}$ and $V_3 = \{(0,1), (1,1)\}$.

Case III. $i_1 = i_2 < i_3$

For this case, define π_1 as a projection map from $[m]^t$ to $\{0,1\}$, $\{0,1,2\}$ or $\{0,1,2,3\}$ depending on the cardinality of $I = \{x_1, y_1\} \cap \{x_2, y_2\}$ and π_2 as a map from $[m]^t$ to $\{0,1,*\}$, similarly to above.

If $I = \emptyset$, then the graph has two disjoint components, one containing edges arising from c_1 and c_3 and the other edges arising from c_2 and c_3. Since both components are formed by the union of two colors, they are 3-colorable and the result follows. The most delicate case is when $|I| = 2$ and $a_{1,i_3} = 0$ and $a_{2,i_3} = 1$ (or vice versa). In this case, the coloring is given by

$$V_1 = \{(0,0)\},\ V_2 = \{(1,0)\},\ V_3 = \{(0,1), (0,*)\},\ V_4 = \{(1,1), (1,*)\}.$$

The other cases can also be checked to be 4-colorable. We omit the details.

Case IV. $i_1 = i_2 = i_3$.

A similar deduction shows that we only need to consider graphs with at most three edges, which are clearly 3-colorable.

See Figure 2 for an illustration.

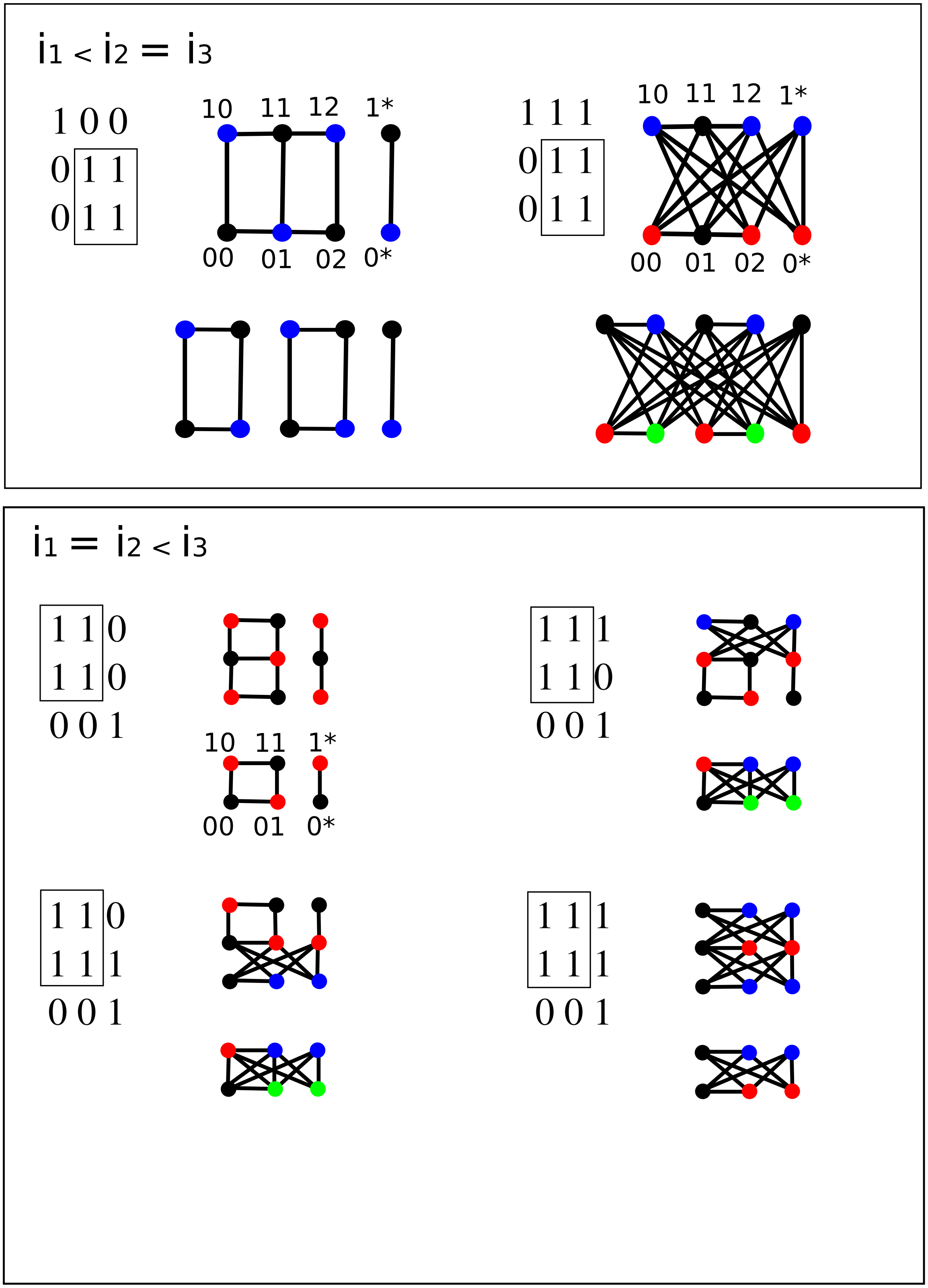

Figure 2: Colorings for Case II and III

References

[1] D. Conlon, J. Fox, C. Lee and B. Sudakov, On the grid Ramsey problem and related questions, *submitted*.

[2] D. Conlon, J. Fox, C. Lee and B. Sudakov, The Erdős-Gyárfás problem on generalized Ramsey numbers, *submitted*.

[3] D. Mubayi, Edge-coloring cliques with three colors on all 4-cliques, *Combinatorica* **18** (1998), 293–296.