Combinatorics, Probability and Computing (2003) 12, 599–620.
DOI: 10.1017/S096354830300587X Printed in the United Kingdom

Rainbow Arithmetic Progressions and Anti-Ramsey Results

VESELIN JUNGIĆ,[1†] JACOB LICHT,[2] MOHAMMAD MAHDIAN,[3]
JAROSLAV NEŠETŘIL[4‡] and RADOŠ RADOIČIĆ[3]

[1]Department of Mathematics, Simon Fraser University, Burnaby, BC, V54 1S6, Canada
(e-mail: vjungic@sfu.ca)

[2]William H. Hall High School, 975 North Main Street, West Hartford, CT 06117, USA
(e-mail: licht@mit.edu)

[3]Department of Mathematics, MIT, Cambridge, MA 02139, USA
(e-mail: {mahdian,rados}@math.mit.edu)

[4]Department of Applied Mathematics, Charles University, Prague, Czech Republic
(e-mail: nesetril@kam.ms.mff.cuni.cz)

Received 23 May 2002; revised 15 June 2003

The van der Waerden theorem in Ramsey theory states that, for every k and t and sufficiently large N, every k-colouring of $[N]$ contains a monochromatic arithmetic progression of length t. Motivated by this result, Radoičić conjectured that every equinumerous 3-colouring of $[3n]$ contains a 3-term rainbow arithmetic progression, *i.e.*, an arithmetic progression whose terms are coloured with distinct colours. In this paper, we prove that every 3-colouring of the set of natural numbers for which each colour class has density more than 1/6, contains a 3-term rainbow arithmetic progression. We also prove similar results for colourings of $\mathbb{Z}_n$. Finally, we give a general perspective on other *anti-Ramsey-type* problems that can be considered.

1. Introduction

In 1916, Schur [29] proved that for every k, if n is sufficiently large, then every k-colouring of $[n] := \{1, \ldots, n\}$ contains a monochromatic solution of the equation $x + y = z$. More than seven decades later, Alekseev and Savchev [1] considered what Bill Sands calls an *un-Schur* problem [15]. They proved that for every equinumerous 3-colouring of $[3n]$ (*i.e.*, a colouring in which different colour classes have the same cardinality), the equation

† Supported by the Czech Research Grant GA\V, CR/201/99/0242.
‡ Supported by Project LN00A056 of the Czech Ministry of Education.

$x + y = z$ has a solution with x, y and z belonging to different colour classes. Such solutions will be called *rainbow* solutions. E. and G. Szekeres asked whether the condition of equal cardinalities for three colour classes can be weakened [31]. Indeed, Schönheim [28] proved that for every 3-colouring of $[n]$, such that every colour class has cardinality greater than $n/4$, the equation $x + y = z$ has rainbow solutions. Moreover, he showed that $n/4$ is optimal.

Inspired by the problem above, Radoičić posed the following conjecture at the open problem session of the MIT Combinatorics Seminar.

Conjecture 1.1. *For every equinumerous* 3*-colouring of* $[3n]$*, there exists a rainbow* $AP(3)$*, i.e., a solution to the equation* $x + y = 2z$ *in which* x*,* y*, and* z *are coloured with three different colours.*

This conjecture can be considered as the counterpart of van der Waerden's theorem in Ramsey theory. Van der Waerden's theorem states that, for every k and t, if N is sufficiently large, then every k-colouring of $[N]$ contains a monochromatic t-term arithmetic progression.

Backed by the computer evidence ($n \leqslant 56$), we pose the following stronger form of Conjecture 1.1.

Conjecture 1.2. *For every* $n \geqslant 3$*, every partition of* $[n]$ *into three colour classes* $\mathscr{R}$*,* $\mathscr{G}$*, and* $\mathscr{B}$ *with* $\min(|\mathscr{R}|, |\mathscr{G}|, |\mathscr{B}|) > r(n)$*, where*

$$r(n) := \begin{cases} \lfloor (n+2)/6 \rfloor & \text{if } n \not\equiv 2 \pmod 6, \\ (n+4)/6 & \text{if } n \equiv 2 \pmod 6, \end{cases} \tag{1.1}$$

contains a rainbow $AP(3)$.

Throughout the paper, R, G and B correspond to the colours red, green and blue, respectively.

Unable to settle the above conjectures, in this paper we prove the following *infinite* version of Conjecture 1.2.

Theorem 1.3. *Every* 3*-colouring of the set of natural numbers* $\mathbb{N}$ *with the upper density of each colour greater than* $1/6$ *contains a rainbow* $AP(3)$.

A more precise statement of the above theorem and its proof will be presented in Section 2. We also show that there exists a 3-colouring of $[n]$ with $\min(|\mathscr{R}|, |\mathscr{G}|, |\mathscr{B}|) = r(n)$, where r is the function defined in (1.1), that contains no rainbow $AP(3)$. This shows that Conjecture 1.2, if true, is the best possible.

An interesting corollary of Theorem 1.3 is the *modular* version of Conjecture 1.2, which states that if $\mathbb{Z}_n$ is coloured with three colours such that the size of every colour class is greater than $n/6$, then there exist x, y and z, each of a different colour with $x + y \equiv 2z \pmod n$. It turns out that in this case $n/6$ is not the best possible. We will discuss further generalizations of the modular case of Conjecture 1.2 in Section 3.

Previous work regarding the existence of rainbow structures in a coloured universe has been done in the context of *canonical* Ramsey theory (see [9, 8, 7, 25, 24, 18, 19, 20, 26] and references therein). However, the canonical theorems prove the existence of *either* a monochromatic structure *or* a rainbow structure. Our results are not 'either–or' statements and are thus the first results in the literature guaranteeing the existence of rainbow arithmetic progressions. In a sense, the conjectures and theorems above can be thought of as the first rainbow counterparts of classical theorems in Ramsey theory, such as van der Waerden's, Rado's and Szemerédi's theorems [14]. It is curious to note that anti-Ramsey problems have received great attention in the context of graph theory as well (see [10, 6, 2, 3, 27, 11, 5, 22, 17, 4, 21] and references therein).

In Section 4, we present a Rado-type theorem for colourings of $\mathbb{Z}_p$, using both classical and recent results from additive number theory. Finally, in Section 5, we give several open problems and a general perspective of various research problems in this area.

2. The infinite form of our conjecture

Assume $c : \mathbb{N} \mapsto \{R, G, B\}$ is a 3-colouring of the set of natural numbers with colours red, green, and blue. We can also think of c as an infinite sequence of the elements of $\{R, G, B\}$. Let $\mathscr{R}_c(n)$ be the number of integers less than or equal to n that are coloured red. In other words, $\mathscr{R}_c(n) := |[n] \cap \{i : c(i) = R\}|$. $\mathscr{G}_c(n)$ and $\mathscr{B}_c(n)$ are defined similarly. A *rainbow* $AP(3)$ is a sequence a_1, a_2, a_3 such that $a_1 + a_3 = 2a_2$ and $c(a_i) \neq c(a_j)$ for every $i \neq j$. We say that c is *rainbow-free* if it does not contain any rainbow $AP(3)$.

Theorem 2.1. *Let c be a 3-colouring of $\mathbb{N}$ such that*

$$\limsup_{n\to\infty} \left(\min(\mathscr{R}_c(n), \mathscr{G}_c(n), \mathscr{B}_c(n)) - n/6\right) = +\infty. \tag{2.1}$$

Then c contains a rainbow $AP(3)$.

Before proving Theorem 2.1 we define a few terms. We say that a string $s = (s_1, \ldots, s_k) \in \{R, G, B, ?\}^k$ appears in c if there exists an i such that, for every $j = 1, \ldots, k$, either $s_j = c(i + j)$ or $s_j = ?$. In this case, s appears in c at position i. For $i_1, i_2 \in \mathbb{N}$, $i_1 < i_2 - 1$, and $\{x, y, z\} = \{R, G, B\}$, we say that c has a *colour-change* of type xyz at positions (i_1, i_2), if $c(i_1) = x$, $c(i_2) = z$, and $c(j) = y$ for every $i_1 < j < i_2$.

Lemma 2.2. *Let c be a rainbow-free 3-colouring of $\mathbb{N}$. If there is a colour-change of type xyz at position (i_1, i_2) for some $1 < i_1 < i_2 - 1$, then $c(i_1 - 1) = c(i_2 + 1) = y$.*

Proof. If $c(i_1 - 1) = z$, then $i_1 - 1, i_1, i_1 + 1$ is a rainbow $AP(3)$. Therefore, $c(i_1 - 1)$ is either y or x. Assume $c(i_1 - 1) = x$. One of the numbers $i_1 - 1$ and i_1 has the same parity as i_2. Let i_1' denote this number. It is easy to see that $i_1', (i_1' + i_2)/2, i_2$ is a rainbow $AP(3)$. This contradiction shows that $c(i_1 - 1) = y$. Similarly, $c(i_2 + 1) = y$. □

Corollary 2.3. *Let c be a rainbow-free 3-colouring of $\mathbb{N}$. If there is a colour-change of type xyz at position (i_1, i_2) for some $1 < i_1 < i_2$, then both $yxyy?y$ and $y?yyzy$ appear in c at positions $i_1 - 1$ and $i_2 - 4$, respectively.*

Proof. It suffices to note that if c has a colour-change at position (i_1, i_2), then $i_1 - i_2$ is odd, for otherwise $i_1, (i_1 + i_2)/2, i_2$ is a rainbow $AP(3)$. This, together with Lemma 2.2, implies that if there is a colour-change of type xyz at position (i_1, i_2), then $c(i_1 + 4) = c(i_2 - 4) = y$. □

Lemma 2.4. *Every 3-colouring of $\mathbb{N}$ that contains both a colour-change of type xyz and a colour-change of type xzy contains a rainbow $AP(3)$.*

Proof. Assume c is a 3-colouring of $\mathbb{N}$ that contains a colour-change of type xyz at position (i_1, i_2) and a colour-change of type xzy at position (i'_1, i'_2). By Corollary 2.3, c contains $yxyy?y$ and $zxzz?z$ at positions $i_1 - 1$ and $i'_1 - 1$. Consider the following two cases.

Case 1: $i_1 \equiv i'_1 \pmod 2$.
In this case, consider one of the following arithmetic progressions based on the value of $c((i_1 + i'_1 + 2)/2)$:

$$\begin{array}{rccl} i_1+1, & (i_1+i'_1+2)/2, & i'_1+1 & \text{if } c((i_1+i'_1+2)/2) = x, \\ i_1, & (i_1+i'_1+2)/2, & i'_1+2 & \text{if } c((i_1+i'_1+2)/2) = y, \\ i_1+2, & (i_1+i'_1+2)/2, & i'_1 & \text{if } c((i_1+i'_1+2)/2) = z. \end{array}$$

Case 2: $i_1 \not\equiv i'_1 \pmod 2$.
In this case, consider one of the following arithmetic progressions based on the value of $c((i_1 + i'_1 + 1)/2)$:

$$\begin{array}{rccl} i_1-1, & (i_1+i'_1+1)/2, & i'_1+2 & \text{if } c((i_1+i'_1+1)/2) = x, \\ i_1, & (i_1+i'_1+1)/2, & i'_1+1 & \text{if } c((i_1+i'_1+1)/2) = y, \\ i_1+1, & (i_1+i'_1+1)/2, & i'_1 & \text{if } c((i_1+i'_1+1)/2) = z. \end{array}$$

It is easy to see that in each case the arithmetic progression that we considered is a rainbow arithmetic progression. □

Similarly, we can prove that a rainbow-free 3-colouring of $\mathbb{N}$ cannot contain colour-changes of type xyz and yxz at the same time. Therefore, we get the following corollary.

Corollary 2.5. *Let c be a rainbow-free 3-colouring of $\mathbb{N}$. Then, for every two types of colour-changes that are connected in Figure 1 by an edge, c cannot contain both of them.*

The following lemma shows an important property of rainbow-free 3-colourings of $\mathbb{N}$. Note that we do not need any assumption about the density of colours here. In fact, it

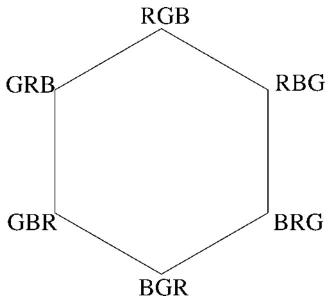

Figure 1. Different types of colour-changes

is possible to prove the conclusion of this lemma even without the assumption that each colour is used infinitely many times.

Lemma 2.6. *Let c be a rainbow-free 3-colouring of $\mathbb{N}$. Assume each colour is used for colouring infinitely many numbers in c. Then there are two distinct colours $x, y \in \{R, G, B\}$ that never appear next to each other in c.*

Proof. Assume, for a contradiction, that every two distinct colours appear next to each other somewhere in c. In other words, for any two distinct colours x and y, there is an i such that one of i and $i+1$ is coloured with x and the other is coloured with y. Consider the smallest number j greater than i that is coloured with the third colour, z. Such a number exists, since by assumption each colour is used infinitely often in c. There must be a colour-change of type xyz or yxz at position (j', j), for some $j' < j$. This shows that, for every three distinct colours $x, y, z \in \{R, G, B\}$, either a colour-change of type xyz, or a colour-change of type yxz must appear in c. This together with Corollary 2.5 implies that, for every two types of colour-changes that are connected in Figure 1 by an edge, c contains exactly one of them. Therefore, either c contains colour-changes of types RGB, BRG, and GBR, and no colour-change of type RBG, BGR, or GRB, or *vice versa.* We assume, without loss of generality, that c contains colour-changes of types RGB, BRG, and GBR, and does not contain any colour-change of type RBG, BGR, or GRB.

Consider a colour-change of type RGB at position (i_1, i_2). Let i_4 be the smallest number greater than i_2 that is coloured red, and let i_6 be the smallest number greater than i_4 that is coloured green. Since c does not contain any colour-change of type BGR or RBG, there must be a colour-change of type GBR at position (i_3, i_4) for some $i_2 < i_3 < i_4$, and a colour-change of type BRG at position (i_5, i_6) for some $i_4 < i_5 < i_6$. Notice that all numbers between i_2 and i_3 are coloured blue or green, and all numbers between i_4 and i_5 are coloured blue or red (see Figure 2). One important observation is that R and G do not appear next to each other after i_1 and before i_6.

By Corollary 2.3, c contains $G?GGBG$ and $RBRR?R$ at positions $i_2 - 4$ and $i_5 - 1$. We consider two cases based on the parity of $i_2 + i_5$.

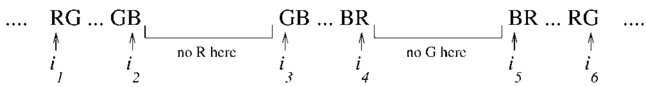

Figure 2. Lemma 2.6

Case 1: $i_2 + i_5$ is *odd*.
Consider the number $(i_2 + i_5 - 1)/2$. In c, this number cannot be coloured red, for otherwise we have a rainbow $AP(3)$: $i_2 - 1, (i_2 + i_5 - 1)/2, i_5$. Also, it cannot be coloured blue because of the arithmetic progression $i_2 - 2, (i_2 + i_5 - 1)/2, i_5 + 1$. Therefore, $c((i_2 + i_5 - 1)/2) = G$. Similarly, the arithmetic progressions $i_2, (i_2 + i_5 + 1)/2, i_5 + 1$ and $i_2 - 1$, $(i_2 + i_5 + 1)/2, i_5 + 2$ show that $c((i_2 + i_5 + 1)/2) = R$. But this is in contradiction to the observation that G and R never appear next to each other between i_1 and i_6.

Case 2: $i_2 + i_5$ is *even*.
Considering the arithmetic progressions $i_2 - 2, (i_2 + i_5 - 2)/2, i_5$ and $i_2 - 1, (i_2 + i_5 - 2)/2$, $i_5 - 1$ shows that $c((i_2 + i_5 - 2)/2) = G$. Also, $c((i_2 + i_5 + 2)/2) = R$ because of the arithmetic progressions $i_2, (i_2 + i_5 + 2)/2, i_5 + 2$ and $i_2 + 1, (i_2 + i_5 + 2)/2, i_5 + 1$. Since G and R never appear next to each other between i_1 and i_6, $(i_2 + i_5)/2$ cannot be coloured with green or red. Therefore it is coloured blue. Thus, $(i_2 + i_5 - 2)/2), (i_2 + i_5)/2, (i_2 + i_5 + 2)/2$ is a rainbow $AP(3)$, which is a contradiction.

Therefore, the assumption that every two distinct colours appear next to each other leads to a contradiction in both cases. □

Lemma 2.6 shows that, for any rainbow-free 3-colouring, there is a colour z such that, for every two consecutive numbers that are coloured with different colours, at least one of them is coloured with z. We call such a colour a *dominant colour*. In the rest of this proof, we assume, without loss of generality, that red is the dominant colour. In other words, we will assume that B and G do not appear next to each other in c.

Lemma 2.7. *Let c be a rainbow-free 3-colouring of $\mathbb{N}$ and assume red is the dominant colour in c. Then there is a position i and a colour $x \in \{B, G\}$ such that there are no two consecutive xs after position i.*

Proof. Assume, for a contradiction, that c is a 3-colouring of $\mathbb{N}$ with no rainbow $AP(3)$ in which BB and GG appear infinitely many times, and R is the dominant colour. Therefore there is an $i_1 < i_2 < i_3$, such that BB appears at positions i_1 and i_3, and GG appears at position i_2. Let j_1 be the largest number less than i_2 such that a BB appears at position j_1, and let j_2 be the smallest number greater than i_2 such that a BB appears at position j_2. Let $k_1, k_1 + 1, \ldots, k_2$ be the longest sequence of consecutive numbers between j_1 and j_2 that are coloured green (*i.e.*, $j_1 < k_1 < k_2 < j_2$, $c(k) = G$ for every $k_1 \leqslant k \leqslant k_2$, and $k_2 - k_1 + 1$ is maximum). By the definition of j_1 and j_2, neither $j_1 + 2$ nor $j_2 - 1$ is coloured blue. Therefore, since red is the dominant colour, $c(j_1 + 2) = c(j_2 - 1) = R$. Consider one of the numbers j_1 or $j_1 + 1$ that has the same parity as $j_2 - 1$. The arithmetic progression consisting of this number, $j_2 - 1$, and their midpoint $\lfloor (j_1 + j_2)/2 \rfloor$ shows that

$c(\lfloor (j_1+j_2)/2 \rfloor) \neq G$. Similarly, the red at j_1+2 and one of the blues at j_2 or j_2+1 imply that $c(\lceil (j_1+j_2)/2 \rceil + 1) \neq G$. Therefore, since $k_1 < k_2$, we have either $k_2 < \lfloor (j_1+j_2)/2 \rfloor$ or $k_1 > \lceil (j_1+j_2)/2 \rceil + 1$.

Assume $k_2 < \lfloor (j_1+j_2)/2 \rfloor$. For every i, $k_1 \leqslant i \leqslant k_2$, the arithmetic progressions $j_1, i, 2i - j_1$ and $j_1+1, i, 2i-j_1-1$ show that $2i-j_1-1$ and $2i-j_1$ are not coloured red. Therefore, none of the numbers between $2k_1-j_1-1$ and $2k_2-j_1$ is red. This, together with the fact that red is the dominant colour, implies that all of the numbers between $2k_1-j_1-1$ and $2k_2-j_1$ must be coloured with the same colour, either blue or green. If they are all blue, we get a contradiction to the definition of j_1 and j_2, as these definitions imply that no BB appears after j_1 and before j_2. If they are all green, we have a contradiction to the definition of k_1 and k_2, since by the assumption $k_2 < \lfloor (j_1+j_2)/2 \rfloor$, the sequence $2k_1-j_1-1, \ldots, 2k_2-j_1$ is a sequence of greens between j_1 and j_2 that is longer than the sequence $k_1, \ldots, k_2$.

Therefore we get a contradiction in either case. A symmetric argument leads to a similar contradiction for the case $k_1 > \lceil (j_1+j_2)/2 \rceil + 1$. □

Next we show that the density assumption (2.1) implies that the dominant colour must appear in c with a high frequency. We start with the following simple lemma.

Lemma 2.8. *Let c be a 3-colouring of $\mathbb{N}$ that satisfies the density assumption* (2.1). *Then there is a $k \leqslant 5$ such that, for every i, there exists $j > i$ such that j and $j+k$ are both coloured green.*

Proof. Assume not; then there is an i such that every two numbers greater than i that are coloured green are at least 6 apart. Therefore, $\mathcal{G}_c(n) \leqslant n/6 + i$, which contradicts (2.1). □

Lemma 2.9. *If c is a rainbow-free 3-colouring of $\mathbb{N}$ that satisfies the density assumption* (2.1), *and red is a dominant colour in c, then there is n_0 such that, for every $i > n_0$, either $c(i)$ or $c(i+1)$ is red.*

Proof. By Lemma 2.7, the number of appearances of either BB or GG in c is finite. Assume, without loss of generality, that GG appears only a finite number of times in c. That is, there is an n_0 such that no GG appears in c after n_0. If no BB appears after n_0, then we are done. Otherwise, consider a BB at position $i > n_0$.

By Lemma 2.8, there exists a $k \leqslant 5$ and $j > i$ such that j and $j+k$ are both coloured green. The arithmetic progressions $i, j, 2j-i$ and $i+1, j, 2j-i-1$ imply that $2j-i-1$ and $2j-i$ are not red. Therefore, since red is the dominant colour, they are either both blue or both green. The latter case is impossible, since $2j-i-1 > n_0$. This shows that there is a BB at position $2j-i-1$. Similarly, having a BB at position $2j-i-1$ and a G at position $j+k$ implies that there is another BB at position $i+2k$ (see Figure 3).

Repeating the same argument, we conclude that BB appears at positions $i+2kt$ for every integer $t \geqslant 0$. Using Lemma 2.2 it is not difficult to see that if there is a BB at position i_1, and a G at position $i_2 > i_1$, then $i_2 \geqslant i_1 + 6$. Similarly, if there is a BB at

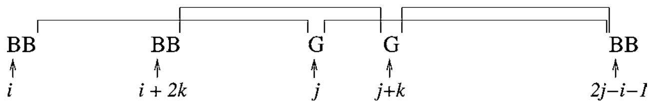

Figure 3. Lemma 2.9

position i_1 and a G at position $i_2 < i_1$, then $i_2 \leqslant i_1 - 5$. Since $k \leqslant 5$, these facts imply that for every $t \geqslant 0$, none of the numbers between $i + 2kt$ and $i + 2k(t+1)$ is coloured green. Therefore, the number of greens is finite, which is a contradiction. □

Lemma 2.10. *If c be a rainbow-free 3-colouring of $\mathbb{N}$ that satisfies the density assumption (2.1), and red is a dominant colour in c, then there is an n_0 such that, for every $i > n_0$, either $c(i)$ or $c(i+2)$ is red.*

Proof. By Lemma 2.9, there is an n_0 such that, for every $i > n_0$, either i or $i+1$ is coloured red. Assume, for a contradiction, that there exists $i > n_0$ such that neither i nor $i+2$ is coloured red. By Lemma 2.9, $c(i+1) = R$. Therefore, i and $i+2$ are either both green, or both blue. Assume, without loss of generality, that they are both blue. Consider an arbitrary $l > i$ whose parity is the same as the parity of i. If l is coloured green, then the arithmetic progressions $i, (i+l)/2, l$ and $i+2, (i+l)/2+1, l$ show that neither $(i+l)/2$ nor $(i+l)/2+1$ is red, which contradicts Lemma 2.9. Therefore, no $l > i$ with the same parity as i is coloured green.

Now consider an arbitrary $i' \geqslant i$ that is coloured blue and has the same parity as i. Using Lemma 2.8, there is a $j > i'$ such that j and $j+k$ are both coloured green (for a fixed $k \leqslant 5$). By the above argument, neither j nor $j+k$ has the same parity as i. Therefore, k is either 2 or 4. The arithmetic progression $i', j, 2j - i'$ shows that $2j - i'$ is not red. Also, since it has the same parity as i, it cannot be green. Therefore, $c(2j - i') = B$. Similarly, the arithmetic progression $i' + 2k, j+k, 2j - i'$ and the fact that $i' + 2k$ has the same parity as i show that $c(i' + 2k) = B$. This means that, for every $i' > i$ with the same parity as i, if i' is coloured blue, then so is $i' + 2k$. Thus, all numbers $i + 2kt$ and $i + 2kt + 2$ for $t \geqslant 0$ must be coloured blue.

- If $k = 2$, this means that every number greater than i that has the same parity as i is coloured blue. Therefore, by Lemma 2.6, no number greater than i is coloured green, which is a contradiction.
- If $k = 4$, this means that for every integer $t \geqslant 0$, $i + 8t$, $i + 8t + 2$ and $i + 8t + 8$ are coloured blue. Therefore, by Lemma 2.6, $i + 8t + 1$, $i + 8t + 3$, and $i + 8t + 7$ are not green. Also, $i + 8t + 4$ and $i + 8t + 6$ have the same parity as i and therefore cannot be coloured green. Thus, the only numbers that can be coloured green are of the form $i + 8t + 5$. Therefore, $\mathcal{G}_c(n) \leqslant n_0 + \frac{1}{8}n$, which contradicts (2.1). □

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Assume c does not contain any rainbow $AP(3)$. Therefore, by Lemma 2.6, there is a dominant colour. Assume without loss of generality that the dominant colour is red. By Lemmas 2.9 and 2.10 there is an n_0 such that, for every $i > n_0$, at least two of the numbers $i, i+1, i+2$ are coloured red. Therefore, for every n, $\mathscr{R}_c(n) \geqslant \frac{2}{3}(n - n_0)$. Thus, $\min(\mathscr{G}_c(n), \mathscr{B}_c(n)) \leqslant \frac{1}{2}(n - \frac{2}{3}(n - n_0)) = \frac{1}{6}n + \frac{1}{3}n_0$, contradicting (2.1). $\square$

A natural question is whether the assumption (2.1) in Theorem 2.1 can be weakened. Notice that Conjecture 1.2 suggests that the conclusion of Theorem 2.1 is true with the weaker assumption that $\limsup_{n\to\infty}(\min(\mathscr{R}_c(n), \mathscr{G}_c(n), \mathscr{B}_c(n)) - \frac{1}{6}n) > \frac{4}{6}$. We still have not been able to prove this fact. However, the following proposition shows that the constant $1/6$ in the density assumption cannot be replaced with a smaller constant.

Proposition 2.11. *There is a rainbow-free 3-colouring c of $\mathbb{N}$ such that, for every n,*

$$\min(\mathscr{R}_c(n), \mathscr{G}_c(n), \mathscr{B}_c(n)) = \lfloor (n+2)/6 \rfloor.$$

Proof. Consider the following colouring of $\mathbb{N}$:

$$c(i) := \begin{cases} B & \text{if } i \equiv 1 \pmod 6, \\ G & \text{if } i \equiv 4 \pmod 6, \\ R & \text{otherwise.} \end{cases}$$

It is easy to see that c contains no rainbow $AP(3)$ and $\min(\mathscr{R}_c(n), \mathscr{G}_c(n), \mathscr{B}_c(n)) = \mathscr{G}_c(n) = \lfloor (n+2)/6 \rfloor$. $\square$

The following proposition shows that Conjecture 1.2, if true, is the best possible.

Proposition 2.12. *For every $n \geqslant 3$, there is a rainbow-free 3-colouring c of $[n]$ in which the size of the smallest colour class is $r(n)$, where r is the function defined in* (1.1).

Proof. For $n \not\equiv 2 \pmod 6$, Proposition 2.11 gives such a colouring. Assume $n = 6k + 2$ for an integer k. We define a colouring c as follows:

$$c(i) := \begin{cases} B & \text{if } i \leqslant 2k+1 \text{ and } i \text{ is odd}, \\ G & \text{if } i \geqslant 4k+2 \text{ and } i \text{ is even}, \\ R & \text{otherwise.} \end{cases}$$

Since every blue number is at most $2k+1$, and every green number is at least $4k+2$, a blue and a green number cannot be the first and second, or the second and third terms of an arithmetic progression with all terms in $[n]$. Also, since blue numbers are odd and green numbers are even, a blue and a green cannot be the first and third terms of an arithmetic progression. Therefore, c does not contain any rainbow $AP(3)$. It is not difficult to see that c contains no rainbow $AP(3)$ and $\min(\mathscr{R}_c(n), \mathscr{G}_c(n), \mathscr{B}_c(n)) = k + 1 = (n+4)/6$. $\square$

3. Rainbow arithmetic progressions in $\mathbb{Z}_n$

A 3-term arithmetic progression ($AP(3)$) in $\mathbb{Z}_n$ is a sequence a_1, a_2, a_3 such that $a_1 + a_3 \equiv 2a_2 \pmod n$. For a 3-colouring $c : \mathbb{Z}_n \mapsto \{R, G, B\}$ of $\mathbb{Z}_n$, we define $\mathscr{R}_c := \{i : c(i) = R\}$. $\mathscr{G}_c$

and $\mathscr{B}_c$ are defined similarly. Also, from a 3-colouring c of $\mathbb{Z}_n$, we define a 3-colouring $\bar{c}$ of $\mathbb{N}$ as follows: for every $i \in \mathbb{N}$, $\bar{c}(i) := c(i \bmod n)$. An interesting corollary of Theorem 2.1 is the following.

Theorem 3.1. *Every* 3-*colouring* c *of* $\mathbb{Z}_n$ *with* $\min(|\mathscr{R}_c|, |\mathscr{G}_c|, |\mathscr{B}_c|) > n/6$ *contains a rainbow* $AP(3)$.

Proof. Consider the colouring $\bar{c}$ of $\mathbb{N}$ defined above. The assumption $\min(|\mathscr{R}_c|, |\mathscr{G}_c|, |\mathscr{B}_c|) > n/6$ implies that

$$\limsup_{m\to\infty} (\min(\mathscr{R}_{\bar{c}}(m), \mathscr{G}_{\bar{c}}(m), \mathscr{B}_{\bar{c}}(m)) - m/6) = +\infty.$$

Therefore, by Theorem 2.1, there is a rainbow $AP(3)$ in $\bar{c}$. By computing the terms of this arithmetic progression modulo n we obtain a rainbow $AP(3)$ in c. $\square$

A natural question is whether the condition $\min(|\mathscr{R}_c|, |\mathscr{G}_c|, |\mathscr{B}_c|) > n/6$ in Theorem 3.1 can be weakened. For n divisible by 6, the colouring defined in Proposition 2.11 shows that this condition is tight. However, for most other values of n it is possible to use number-theoretic properties of $\mathbb{Z}_n$ to replace this condition with a weaker assumption. The following theorem is an example.

Theorem 3.2. *Let n be an odd number and let q be the smallest prime factor of n. Then every* 3-*colouring* c *of* $\mathbb{Z}_n$ *with* $\min(|\mathscr{R}_c|, |\mathscr{G}_c|, |\mathscr{B}_c|) > n/q$ *contains a rainbow* $AP(3)$.

First, we prove the following lemma.

Lemma 3.3. *Let c be a* 3-*colouring of* $\mathbb{Z}_n$, *let a be an integer relatively prime to n, and let b be an arbitrary integer. Let* $c'(i) := c((ai + b) \bmod n)$ *for every* $i \in \mathbb{Z}_n$. *Then c contains a rainbow* $AP(3)$ *if and only if c' contains a rainbow* $AP(3)$. *Furthermore,* $|\mathscr{R}_{c'}| = |\mathscr{R}_c|$, $|\mathscr{G}_{c'}| = |\mathscr{G}_c|$, *and* $|\mathscr{B}_{c'}| = |\mathscr{B}_c|$.

Proof. It is enough to note that since a is relatively prime to n, the mapping $i \mapsto ai + b \pmod n$ is an automorphism of $(\mathbb{Z}_n, +)$. $\square$

Proof of Theorem 3.2. Assume, for a contradiction, that we have a 3-colouring c of $\mathbb{Z}_n$ with no rainbow $AP(3)$ such that $\min(|\mathscr{R}_c|, |\mathscr{G}_c|, |\mathscr{B}_c|) > n/q$. Assume, without loss of generality, that $|\mathscr{G}_c| = \min(|\mathscr{R}_c|, |\mathscr{G}_c|, |\mathscr{B}_c|)$. Since $|\mathscr{G}_c| > n/q$, there exist $k < q$ and i such that i and $i + k$ are both coloured green. Since $k < q$ and q is the smallest prime factor of n, k is relatively prime to n. Therefore, Lemma 3.3 with $a = k$ and $b = i$ gives a colouring with the same properties as c in which 0 and 1 are both coloured green. From now on, we let c denote this colouring. Therefore, c does not contain any rainbow $AP(3)$, and it satisfies $|\mathscr{G}_c| = \min(|\mathscr{R}_c|, |\mathscr{G}_c|, |\mathscr{B}_c|) > n/q$ and $c(0) = c(1) = G$.

From c, we construct a colouring $\bar{c}$ of $\mathbb{N}$ as in the proof of Theorem 3.1. Lemma 2.6 shows that there is a dominant colour in $\bar{c}$. We consider the following two cases.

Case 1: G is the dominant colour in $\bar{c}$.
Since $\bar{c}$ is periodic, Lemma 2.7 implies that $\bar{c}$ cannot contain BB and RR at the same time. Assume without loss of generality that $\bar{c}$ does not contain any BB. This, together with the fact that G is the dominant colour, implies that in c every B is followed by a G (*i.e.*, for every $i \in \mathbb{Z}_n$, if $c(i) = B$, then $c(i+1) = G$). Furthermore, since by Lemma 2.6 no R can be followed by a B in $\bar{c}$, there must be at least one R in c that is followed by a G. Thus, $|\mathscr{G}_c| \geqslant |\mathscr{B}_c| + 1$, contradicting the assumption that $|\mathscr{G}_c| = \min(|\mathscr{R}_c|, |\mathscr{G}_c|, |\mathscr{B}_c|)$.

Case 2: G is not the dominant colour in $\bar{c}$.
Without loss of generality, assume R is the dominant colour in $\bar{c}$. In $\bar{c}$, GG appears at positions nt for every $t > 0$. Therefore, by Lemma 2.7 no BB appears in c. On the other hand, the assumption $|\mathscr{B}_c| \geqslant n/q$ implies that there exist $k < q$ and i such that i and $i+k$ are both coloured blue. Now, consider the arithmetic progressions $0, i, 2i$ and $1, i, 2i-1$. These arithmetic progressions show that neither of $2i-1$ and $2i$ can be red. Therefore, since c does not contain BB, they must both be green. Similarly, the arithmetic progressions $2k, i+k, 2i$ and $2k+1, i+k, 2i-1$ show that there is a GG at position $2k$. Repeating the same argument implies that there is a GG at position $2kt \pmod n$ for every $t \geqslant 0$. But since k is smaller than the smallest prime factor of n, and n is odd, $2k$ is relatively prime to n. Thus, we have proved that every number in $\{2kt \pmod n) : t \geqslant 0\} = \mathbb{Z}_n$ is coloured green, which is a contradiction. □

For any integer n, we define $m(n)$ as the largest integer m for which there is a rainbow-free 3-colouring c of $\mathbb{Z}_n$ such that $|\mathscr{R}_c|, |\mathscr{G}_c|, |\mathscr{B}_c| \geqslant m$. Theorems 3.1 and 3.2 show that, for every integer n, $m(n) \leqslant \min(n/6, n/q)$, where q is the smallest prime factor of n. Computing the exact value of $m(n)$ for every n remains a challenge. The following theorem gives a general lower bound for the value of $m(n)$.

Theorem 3.4. *Let n be an integer that is not a power of 2 and let q be the smallest odd prime factor of n. Then $m(n) \geqslant \lfloor \frac{n}{2q} \rfloor$.*

Proof. It suffices to show that there is a rainbow-free 3-colouring c of $\mathbb{Z}_n$ satisfying $\min(|\mathscr{R}_c|, |\mathscr{G}_c|, |\mathscr{B}_c|) \geqslant \lfloor \frac{n}{2q} \rfloor$. We know that exactly n/q elements of $\mathbb{Z}_n$ are divisible by q. Colour $\lfloor \frac{n}{2q} \rfloor$ of these numbers with green and the remaining $\lceil \frac{n}{2q} \rceil$ multiples of q with blue. Colour other elements of $\mathbb{Z}_n$ with red. Since q is odd, if two elements of a 3-term arithmetic progression are divisible by q, the third term should also be divisible by q. Therefore, the colouring c constructed above does not contain any rainbow $AP(3)$, and we have $\min(|\mathscr{R}_c|, |\mathscr{G}_c|, |\mathscr{B}_c|) \geqslant \lfloor \frac{n}{2q} \rfloor$. □

In the following theorem we characterize the set of natural numbers n for which $m(n) = 0$.

Theorem 3.5. *For every integer n, there is a rainbow-free 3-colouring of $\mathbb{Z}_n$ with non-empty colour classes if and only if n does* not *satisfy any of the following conditions:*

(a) *n is a power of 2,*
(b) *n is a prime and $\mathrm{ord}_n(2) = n-1$ (i.e., 2 is a generator of $\mathbb{Z}_n$),*
(c) *n is a prime, $\mathrm{ord}_n(2) = (n-1)/2$, and $(n-1)/2$ is an odd number.*

Proof. We first prove the 'if' part. We need to prove that, for every n that does not satisfy any of the above conditions, there is a rainbow-free colouring of $\mathbb{Z}_n$ with no empty colour class. We consider the following two cases: n is not prime, and n is prime.

If n is not a prime number, then by conditions above n can be written as $n = pq$ where p is an odd number and $q > 1$. Let c denote the colouring of $\mathbb{Z}_n$ obtained by colouring 0 with red, other multiples of p with green, and other numbers with blue. In this colouring, every rainbow $AP(3)$ must contain 0 and a multiple of p. Since p is odd, the other term in such an arithmetic progression must also be a multiple of p. Therefore, c is rainbow-free.

If n is a prime number, then we define the colouring c as follows: 0 is coloured with red, all numbers in $\{2^i \bmod n : i \in \mathbb{Z}\} \cup \{-2^i \bmod n : i \in \mathbb{Z}\}$ are coloured with green, and other numbers are coloured with blue. By conditions (b) and (c) we know that either $\mathrm{ord}_n(2) < (n-1)/2$, or $\mathrm{ord}_n(2) = (n-1)/2 = 2k$ for an integer k. In the former case, $|\mathcal{G}_c| \leqslant 2\mathrm{ord}_n(2) < n-1$. In the latter case, we have $2^k = -1$ and therefore $|\mathcal{G}_c| = \mathrm{ord}_n(2) < n-1$. Thus, $\mathcal{B}_c$ is non-empty in either case. Also, every rainbow $AP(3)$ in c must contain 0. Since $\mathcal{G}_c$ is closed under multiplication/division by 2 and -1, any 3-term arithmetic progression that contains 0 and an element of $\mathcal{G}_c$ must contain another element of $\mathcal{G}_c$. Thus, c is rainbow-free.

For the 'only if' part, we need to argue that if n satisfies any of the conditions (a), (b), or (c), then every colouring of $\mathbb{Z}_n$ with non-empty colour classes contains a rainbow $AP(3)$. If n satisfies one of the conditions (b) or (c), then by Theorem 3.2 any colouring c of $\mathbb{Z}_n$ with $\min(|\mathcal{R}_c|, |\mathcal{G}_c|, |\mathcal{B}_c|) > 1$ contains a rainbow $AP(3)$. If $\min(|\mathcal{R}_c|, |\mathcal{G}_c|, |\mathcal{B}_c|) = 1$, then assume without loss of generality that 0 is the only number coloured with red and 1 is coloured with green. For every number $i \in \mathbb{Z}_n \backslash \{0\}$ that is coloured green, $2i$ must also be green; otherwise $0, i, 2i$ will be a rainbow $AP(3)$. Similarly, if i is green, then $-i$ must also be green. This implies that every number in $\{2^i \bmod n : i \in \mathbb{Z}\} \cup \{-2^i \bmod n : i \in \mathbb{Z}\}$ must be coloured green. However, if one of the conditions (b) or (c) hold, then $\{2^i \bmod n : i \in \mathbb{Z}\} \cup \{-2^i \bmod n : i \in \mathbb{Z}\} = \mathbb{Z}_n \backslash \{0\}$. This contradicts the assumption that $\mathcal{B}_c$ is non-empty.

The only case that remains to check is when n satisfies (a), *i.e.*, we need to prove that when $n = 2^k$ for an integer k, there is no rainbow-free colouring of $\mathbb{Z}_n$ with non-empty colour classes. We prove this statement by induction on k. The induction basis is easy to verify. Assume this statement holds for $k-1$, and (for a contradiction) consider a rainbow-free colouring c of $\mathbb{Z}_{2^k}$ with non-empty colour classes.

We can partition $\mathbb{Z}_{2^k}$ into two sets: the set of even numbers $\mathbb{Z}_{2^k}^E = \{2i \bmod 2^k : i \in \mathbb{Z}_{2^k}\}$ and the set of odd numbers $\mathbb{Z}_{2^k}^O = \{(2i+1) \bmod 2^k : i \in \mathbb{Z}_{2^k}\}$. It is clear that each of $\mathbb{Z}_{2^k}^E$ and $\mathbb{Z}_{2^k}^O$ is isomorphic to $\mathbb{Z}_{2^{k-1}}$. Therefore, by the induction hypothesis, if c restricted to either one of them has non-empty colour classes, then c will contain a rainbow $AP(3)$. Thus, we may assume without loss of generality that no element of $\mathbb{Z}_{2^k}^E$ is coloured blue and no element of $\mathbb{Z}_{2^k}^O$ is coloured green. We consider the following two cases.

Case 1: There are two elements g_1, g_2 that are coloured green, and their distance on the circle (*i.e.*, $(g_1 - g_2) \bmod 2^k$) is twice an odd number.
In this case, consider an arbitrary blue element $b \in \mathbb{Z}_{2^k}^O$. Since $(2g_2 - b) \bmod 2^k$ belongs to $\mathbb{Z}_{2^k}^O$, it can not be green. Also, it cannot be red, since otherwise $2g_2 - b, g_2, b$ will be a rainbow $AP(3)$. Thus, for every blue element b, $2g_2 - b$ is also blue. Using the same argument with g_1 and $2g_2 - b$ (instead of g_2 and b), we deduce that for every blue element b, $2g_1 - (2g_2 - b) = b + 2(g_1 - g_2)$ is also blue. Therefore, $b + \gcd(2(g_1 - g_2), 2^k)$ must also be blue. However, from the assumption in this case we know that $\gcd(2(g_1 - g_2), 2^k) = 4$. Thus, in this case, for every blue element b, $b + 4$ is also blue. Now, since there is at least one green element in $\mathbb{Z}_{2^k}^E$, then there exists $b \in \mathbb{Z}_{2^k}^O$ such that $b + 1$ is green. The arithmetic progression $b, b+1, b+2$ and the fact that no element of $\mathbb{Z}_{2^k}^O$ is green imply that $b + 2$ is blue. Similarly, the arithmetic progression $b+1, b+2, b+3$ and the fact that no element of $\mathbb{Z}_{2^k}^E$ is blue imply that $b + 3$ is green. Therefore by induction for every i, $b + 2i$ is blue and $b + (2i + 1)$ is green. This contradicts the assumption that there is at least one red element.

Case 2: For every two green elements g_1 and g_2, their distance is a multiple of 4.
Consider two arbitrary *consecutive* green elements g_1 and g_2, and the set of elements $A = \{g_1 + 1, g_1 + 2, \ldots, g_2\}$ between them in the circular order. Since g_1 and g_2 are consecutive greens, no element of A is green. Thus, all elements of $A \cap \mathbb{Z}_{2^k}^E$ are red. By our assumption in this case $(g_1 + g_2)/2 \in \mathbb{Z}_{2^k}^E$. Therefore, for any element $i \in A \cap \mathbb{Z}_{2^k}^O$, either $i < (g_1 + g_2)/2$ or $i > (g_1 + g_2)/2$. In the first case, the arithmetic progression $(g_1, i, 2i - g_1)$ and the fact that $2i - g_1 \in A \cap \mathbb{Z}_{2^k}^E \subseteq \mathscr{R}_c$ show that i must be red. In the second case, the arithmetic progression $(2i - g_2, i, g_2)$ and the fact that $2i - g_2 \in A \cap \mathbb{Z}_{2^k}^E \subseteq \mathscr{R}_c$ show that i is red. Therefore, for any two consecutive greens g_1 and g_2, all the elements between them are red. This contradicts the assumption that $\mathscr{B}_c$ is nonempty. □

4. Additive number theory and rainbows in $\mathbb{Z}_p$

Strong inverse theorems from additive number theory have proved to be useful tools in Ramsey theory. For example, Gowers' proof of Szemerédi's theorem relies on the theorem of Freiman [13]. Freiman's theorem [12] essentially says that if a set S has small sumset $S + S$, then S is a large subset of a generalized arithmetic progression [23]. Likewise, we will use a recent theorem of Hamidoune and Rødseth [16], generalizing Vosper's classical theorem [33], to prove that *almost* every colouring of $\mathbb{Z}_p$ with three colours has rainbow solutions for *almost* all linear equations in three variables in $\mathbb{Z}_p$. Moreover, we classify all the exceptions.

We write p to denote a prime number and (m, n) to denote the greatest common divisor of m and n. For $b, c \in \mathbb{Z}_p$, we define the set $\{a + di\}_{i=b}^{c}$ in $\mathbb{Z}_p$ as $\{a + di | i \in \mathbb{Z}_p, b \leqslant i \leqslant c\}$, if $b \leqslant c$, and $\{a + di | i \in \mathbb{Z}_p, b \leqslant i \leqslant p - 1 \text{ or } 0 \leqslant i \leqslant c\}$, otherwise. For $X, Y \subset \mathbb{Z}_p$ and $j \in \mathbb{Z}_p$, let $jX = \{jx \mid x \in X\}$, $X - j = \{x - j \mid x \in X\}$ and $X + Y = \{x + y \mid x \in X, y \in Y\}$. If $x \in \mathbb{Z}_p$, let $|k|_p$ denote $\min\{k, p - k\}$. Note that $|k|_p \leqslant \frac{p}{2}$.

Theorem 4.1. *Let $a, b, c, e \in \mathbb{Z}_p$, with $abc \not\equiv 0 \pmod p$. Then every colouring of $\mathbb{Z}_p = \mathscr{R} \cup \mathscr{B} \cup \mathscr{G}$ with $|\mathscr{R}|, |\mathscr{B}|, |\mathscr{G}| \geqslant 4$, contains a rainbow solution of $ax + by + cz \equiv e \pmod p$, with the only exception being the case when $a = b = c =: t$ and every colour class is an arithmetic progression with the same common difference d, so that $d^{-1}\mathscr{R} = \{i\}_{i=a_1}^{a_2-1}$, $d^{-1}\mathscr{B} = \{i\}_{i=a_2}^{a_3-1}$ and $d^{-1}\mathscr{G} = \{i\}_{i=a_3}^{a_1-1}$, where $(a_1 + a_2 + a_3) \equiv t^{-1}e + 1$ or $t^{-1}e + 2 \pmod p$.*

Before proving Theorem 4.1, we recall the classical theorem of Cauchy and Davenport [23] and the recent result of Hamidoune and Rødseth [16].

Theorem (Cauchy–Davenport). *If $S, T \subset \mathbb{Z}_p$, then $|S + T| \geqslant \min\{p, |S| + |T| - 1\}$.*

Theorem (Hamidoune–Rødseth). *Let $S, T \subset \mathbb{Z}_p$, $|S| \geqslant 3$, $|T| \geqslant 3$, $7 \leqslant |S + T| \leqslant p - 4$. Then either $|S + T| \geqslant |S| + |T| + 1$, or S and T are contained in arithmetic progressions with the same common difference and $|S| + 1$ and $|T| + 1$ elements, respectively.*

We also need the following two lemmas.

Lemma 4.2. *If $S \subset \mathbb{Z}_p$ is contained in an arithmetic progression of length $|S| + 1$ with common difference d, then there are at most two pairs of elements of $\mathbb{Z}_p$ of the form $(x, x + d)$ such that $x \in S$ and $x + d \notin S$.*

Proof. Let $S \subset \{a + di\}_{i=0}^{|S|}$. Define $X = \{a + (i + 1)d \mid 0 \leqslant i \leqslant |S|, a + id \in S, a + (i + 1)d \notin S\}$. Then $X \cap S = \emptyset$ and $X \cup S \subset \{a + di\}_{i=0}^{|S|+1}$. Therefore, $|X| + |S| \leqslant |S| + 2$, and $|X| \leqslant 2$. Note that X is precisely the set of elements of the form $x + d$ such that $(x + d) \notin S$ and $x \in S$. $\square$

Lemma 4.3. *Let $p > 7$ and let $S \subset \mathbb{Z}_p$, $3 \leqslant |S| \leqslant p - 5$, be contained in an arithmetic progression of length $|S| + 1$ and common difference d, $(d, p) = 1$. Then every arithmetic progression of length $|S| + 1$ containing S has the common difference equal to d or $p - d$.*

Proof. Let $s := |S|$. Suppose that S is contained in an arithmetic progression of length $s + 1$ and common difference $d' \neq d, p - d$. Applying the group isomorphism $\mathbb{Z}_p \to d^{-1}\mathbb{Z}_p$, we can assume that S is contained in the arithmetic progression $A := \{a + i\}_{i=0}^{s}$, as well as in the arithmetic progression $\bar{A}$ of length $s + 1$ and common difference $\bar{d}$ $(= |d^{-1}d'|_p)$. We have the following three cases.

Case 1: $2 \leqslant \bar{d} \leqslant 4$.
View $\mathbb{Z}_p$ as a circle on p elements and consider the process of looping around the circle in steps of size $\bar{d}$ starting at the first element of $\bar{A}$ and removing the terms of $\bar{A}$ with respect to their order in $\bar{A}$. For $X \subset \mathbb{Z}_p$ and a positive integer i, let $r(i, X)$ be the number of terms removed from a subset X of $\mathbb{Z}_p$ after i loops of this process. Let j be the smallest integer

such that all the terms of $\bar{A}$ have been removed after j loops. It is obvious that $s \leqslant r(j, A)$. Since $\bar{d}$ is relatively prime to p, then an element can be removed from A at most once, and so we have $r(j, A) \leqslant \sum_{i=0}^{j-1} \lceil \frac{s+1-i}{\bar{d}} \rceil$. The number of elements x, $x \in \bar{A}$ and $x \notin A$, removed after j loops is at least $\sum_{i=0}^{j-2} \lfloor \frac{p-(s+1)+i}{\bar{d}} \rfloor$. Hence, $r(j, \bar{A} \backslash A) \geqslant \sum_{i=0}^{j-2} \lfloor \frac{p-(s+1)+i}{\bar{d}} \rfloor$. In the last inequality, the summation goes only up to $j-2$ because removing all the terms of $\bar{A}$ in j loops means only $j-1$ complete loops. It suffices to show that $r(j, \bar{A} \backslash A) > 1$ to get the contradiction to $|\bar{A}| = s+1$.

(a) $\bar{d} = 2$.
$s \leqslant \lceil \frac{s+1}{2} \rceil$ does not hold and, thus, $j \geqslant 2$. Then $r(j, \bar{A} \backslash A) \geqslant \lfloor \frac{p-(s+1)}{2} \rfloor \geqslant 2$.

(b) $\bar{d} = 3$.
$s \leqslant \lceil \frac{s+1}{3} \rceil + \lceil \frac{s}{3} \rceil$ only if $s \in \{3, 4\}$. Hence, either $j = 3$, or $j = 2$ and $s \in \{3, 4\}$. If $j = 3$, then $r(j, \bar{A} \backslash A) \geqslant \lfloor \frac{p-(s+1)}{3} \rfloor + \lfloor \frac{p-s}{3} \rfloor \geqslant 2$. If $j = 2$, then $r(j, \bar{A} \backslash A) \geqslant \lfloor \frac{p-(s+1)}{3} \rfloor \geqslant 2$, since $\leqslant 4$ and $p > 7$.

(c) $\bar{d} = 4$.
If $j \leqslant 2$, then $s \leqslant \lceil \frac{s+1}{4} \rceil + \lceil \frac{s}{4} \rceil$. Hence $s \leqslant 2$, which contradicts $s \geqslant 3$. Therefore, $j \geqslant 3$. Then $r(j, \bar{A} \backslash A) \geqslant \lfloor \frac{p-(s+1)}{4} \rfloor + \lfloor \frac{p-s}{4} \rfloor \geqslant 2$, since $p - s > 4$.

Case 2: $4 < \bar{d} \leqslant |A|$.
Exactly 1 element of A is not in S. Since $\mathbb{Z}_p \backslash S$ has at least 5 elements, no element of $U := \{a+s+1+i\}_{i=0}^{3}$ is in S. Since every element of $U - \bar{d}$ is in A, $U - \bar{d}$ contains at least 3 elements of S. This contradicts Lemma 4.2, because there are 3 pairs $(x, x+\bar{d})$, $x \in S$, $x + \bar{d} \notin S$.

Case 3: $\bar{d} > |A|$.
No element of $V := \{a+i+\bar{d}\}_{i=0}^{3}$ is in S because $a + \bar{d} > a+s+1$ and $a + \bar{d} + 3 \leqslant a + \frac{p}{2} + 3 < a + p$. However, every element of $V - \bar{d}$ is in A. Thus, at least 3 elements of V are in S. This contradicts Lemma 4.2, because there are 3 pairs of elements $(x, x+\bar{d})$, $x \in S$, $x + \bar{d} \notin S$. □

Proof of Theorem 4.1. Assume that there exist $a, b, c, e \in \mathbb{Z}_p$, with $abc \not\equiv 0 \pmod{p}$, and a colouring $\tilde{c}_p$ of $\mathbb{Z}_p = \mathscr{R} \cup \mathscr{B} \cup \mathscr{G}$ into 3 colour classes ($|\mathscr{R}|, |\mathscr{B}|, |\mathscr{G}| \geqslant 4$), containing no rainbow solution of $ax + by + cz \equiv e \pmod{p}$. Let $\mathscr{R}', \mathscr{B}', \mathscr{G}'$ be a permutation of $\mathscr{R}, \mathscr{B}, \mathscr{G}$, and let a', b', c' be a permutation of a, b, c. Since $a'b'c' = abc \not\equiv 0 \pmod{p}$, $|a'\mathscr{R}'| = |\mathscr{R}'|, |b'\mathscr{B}'| = |\mathscr{B}'|, |c'\mathscr{G}'| = |\mathscr{G}'|$. If $|a'\mathscr{R}' + b'\mathscr{B}'| \geqslant |a'\mathscr{R}'| + |b'\mathscr{B}'| + 1$, then by the theorem of Cauchy and Davenport and $|\mathscr{R}'| + |\mathscr{B}'| + |\mathscr{G}'| = p$, $|a'\mathscr{R}' + b'\mathscr{B}' + c'\mathscr{G}'| \geqslant \min\{p, (|\mathscr{R}'| + |\mathscr{B}'| + 1) + |\mathscr{G}'| - 1\} = p$. Hence, there exists a rainbow solution of $ax + by + cz \equiv e$ (mod p), which is a contradiction. Therefore,

$$\begin{aligned}
|a'\mathscr{R}' + b'\mathscr{B}'| &< |a'\mathscr{R}'| + |b'\mathscr{B}'| + 1 = |\mathscr{R}'| + |\mathscr{B}'| + 1 < p - 3, \\
|b'\mathscr{B}' + c'\mathscr{G}'| &< |b'\mathscr{B}'| + |c'\mathscr{G}'| + 1 = |\mathscr{B}'| + |\mathscr{G}'| + 1 < p - 3, \\
|a'\mathscr{R}' + c'\mathscr{G}'| &< |a'\mathscr{R}'| + |c'\mathscr{G}'| + 1 = |\mathscr{R}'| + |\mathscr{G}'| + 1 < p - 3.
\end{aligned}$$

Moreover, using the condition $|\mathscr{R}|, |\mathscr{B}|, |\mathscr{G}| \geqslant 4$ and the theorem of Cauchy and Davenport, we obtain

$$|a'\mathscr{R}' + b'\mathscr{B}'| \geqslant 7, |b'\mathscr{B}' + c'\mathscr{G}'| \geqslant 7, |a'\mathscr{R}' + c'\mathscr{G}'| \geqslant 7.$$

Hence, for every $X, Y \in \{\mathscr{R}, \mathscr{B}, \mathscr{G}\}, X \neq Y$, and every $x, y \in \{a, b, c\}, x \neq y$, we can apply the theorem of Hamidoune and Rødseth on the sets xX and yY; that is, xX and yY are contained in arithmetic progressions with the same common difference and $|X|+1$ and $|Y|+1$ elements, respectively.

The set xX is contained in an arithmetic progression of length $|X|+1$ if and only if X is contained in an arithmetic progression of length $|X|+1$. Thus, $\mathscr{R}$, $\mathscr{B}$ and $\mathscr{G}$ are contained in arithmetic progressions of lengths $|\mathscr{R}|+1$, $|\mathscr{B}|+1$ and $|\mathscr{G}|+1$, respectively. Since every arithmetic progression in $\mathbb{Z}_p$ of common difference d is also an arithmetic progression of common difference $p-d$, Lemma 4.3 implies that there exist unique common differences d_R, d_B and d_G ($\leqslant \frac{p}{2}$) for all arithmetic progressions of lengths $|R|+1$, $|\mathscr{B}|+1$ and $|\mathscr{G}|+1$, containing $\mathscr{R}$, $\mathscr{B}$ and $\mathscr{G}$, respectively.

Let $X, Y \in \{\mathscr{R}, \mathscr{B}, \mathscr{G}\}, X \neq Y$, and $x, y \in \{a, b, c\}, x \neq y$. Since xX and yY are contained in arithmetic progressions with the same common difference and $|X|+1$ and $|Y|+1$ elements, respectively, $y^{-1}xX$ and Y are contained in arithmetic progressions with the common difference d_Y and $|X|+1$ and $|Y|+1$ elements, respectively. Hence, $y^{-1}xd_X = d_Y$.

Similarly, yX and xY are contained in arithmetic progressions with the same common difference and $|X|+1$ and $|Y|+1$ elements, respectively. Thus, $x^{-1}yX$ and Y are contained in arithmetic progressions with the common difference d_Y and $|X|+1$ and $|Y|+1$ elements, respectively. Hence, $x^{-1}yd_X = d_Y$.

It follows that $y^{-1}xd_X = x^{-1}yd_X$, that is, $|x|_p = |y|_p$. Substituting this back into $y^{-1}xd_X = x^{-1}yd_X$, we get $|d_X|_p = |d_Y|_p$. Since all common differences are assumed to be between 0 and $\frac{p}{2}$, we conclude $d_X = d_Y$. This implies $a = b = c =: t$ and $d_R = d_B = d_G =: d$.

Therefore, we can assume that the equation $ax + by + cz \equiv e \pmod{p}$ is of the form $x + y + z \equiv t^{-1}e \pmod{p}$. Since $d = d_R = d_B = d_G$, after applying the group isomorphism $\mathbb{Z}_p \to d^{-1}\mathbb{Z}_p$, we can assume without loss of generality that $\mathscr{R}$, $\mathscr{B}$ and $\mathscr{G}$ are contained in strings of $|\mathscr{R}|+1$, $|\mathscr{B}|+1$ and $|\mathscr{G}|+1$ consecutive elements, respectively. One of the following two cases occurs.

Case 1: There exist at least two colour classes, say $\mathscr{R}$ and $\mathscr{B}$, that are not contained in strings of $|\mathscr{R}|$ and $|\mathscr{B}|$ consecutive elements, respectively.
Then $\mathscr{R} = \{a_1 + i\}_{i=0}^{|\mathscr{R}|-2} \cup \{a_1 + |\mathscr{R}|\}$ and $\mathscr{B} = \{a_1 + |\mathscr{R}| - 1\} \cup \{a_1 + |\mathscr{R}| + i\}_{i=1}^{|\mathscr{B}|-1}$. Then $\mathscr{R} + \mathscr{B} = \{2a_1 + |\mathscr{R}| + i\}_{i=-1}^{|\mathscr{R}|+|\mathscr{B}|-1}$, so that $|\mathscr{R} + \mathscr{B}| = |\mathscr{R}| + |\mathscr{B}| + 1$. By the theorem of Cauchy and Davenport, $|\mathscr{R} + \mathscr{B} + \mathscr{G}| = p$, which implies that the equation $x + y + z \equiv t^{-1}e \pmod{p}$ has a rainbow solution. This case is impossible.

Case 2: $\mathscr{R}$, $\mathscr{B}$ and $\mathscr{G}$ are contained in strings of $|\mathscr{R}|$, $|\mathscr{B}|$ and $|\mathscr{G}|$ consecutive elements, respectively.
Then $\mathscr{R} = \{i\}_{i=a_1}^{a_2-1}$, $\mathscr{B} = \{i\}_{i=a_2}^{a_3-1}$, $\mathscr{G} = \{i\}_{i=a_3}^{a_1-1}$, in which case $\mathscr{R} + \mathscr{B} + \mathscr{G} = \{i\}_{i=a_1+a_2+a_3}^{a_1+a_2+a_3-3}$. Clearly, if there is no rainbow solution to the equation $x + y + z \equiv t^{-1}e \pmod{p}$, then $a_1 + a_2 + a_3 \equiv t^{-1}e + 1$ or $t^{-1}e + 2 \pmod{p}$.

Therefore, if the equation $ax + by + cz \equiv e \pmod{p}$ has no rainbow solutions, then $a = b = c$ and every colour class is an arithmetic progression with the same common difference d, so that $d^{-1}\mathscr{R} = \{i\}_{i=a_1}^{a_2-1}$, $d^{-1}\mathscr{B} = \{i\}_{i=a_2}^{a_3-1}$ and $d^{-1}\mathscr{G} = \{i\}_{i=a_3}^{a_1-1}$, where $(a_1 + a_2 + a_3) \equiv t^{-1}e + 1$ or $t^{-1}e + 2 \pmod{p}$. $\square$

5. Future directions

The problems and conjectures stated in the previous sections deal with the existence of rainbow structures in the sets of integers modulo n. There are many more directions and generalizations we might consider.

One natural direction is generalizing the problems above for rainbow solutions of any linear equation, imitating Rado's theorem about the monochromatic analogue. We have already shown an example of this in Theorem 4.1.

A search for a rainbow counterpart of the Hales–Jewett theorem, though an exciting possibility, led us to some negative results. First, recall some notation from [14]. Define C_t^n, the n-cube over t elements by $C_t^n = \{(x_1, \dots, x_n) : x_i \in \{0, 1, \dots, t-1\}\}$. A *geometric line* in C_t^n is a set of (suitably ordered) points $\mathbf{x_0}, \dots, \mathbf{x_{t-1}}$, $\mathbf{x_i} = (x_{i,1}, \dots, x_{i,n})$ so that in each coordinate j, $1 \leqslant j \leqslant n$, either $x_{0,j} = x_{1,j} = \cdots = x_{t-1,j}$, or $x_{s,j} = s$ for every $0 \leqslant s < t$, or $x_{s,j} = n - s$ for every $0 \leqslant s < t$. The Hales–Jewett theorem implies that, for every t and k, if n is sufficiently large, every k-colouring of C_t^n contains a monochromatic geometric line. This motivates the following question: Is it true that for every equinumerous t-colouring of C_t^n there exists a rainbow geometric line? The following colouring shows that the answer is negative even for small values of t and n. A 3-colouring of C_3^3 defined by

$$\begin{aligned} C_1 &= \{000, 002, 020, 200, 220, 022, 202, 222, 001\}, \\ C_2 &= \{011, 021, 101, 201, 111, 221, 010, 210, 012\}, \\ C_3 &= \{100, 110, 120, 121, 211, 102, 112, 122, 212\} \end{aligned}$$

(parentheses and commas being removed for clarity), has no rainbow geometric lines. Indeed, suppose that $\mathbf{x_0}$, $\mathbf{x_1}$, $\mathbf{x_2}$ is a rainbow geometric line. Suppose that $\mathbf{x_0}$ is coloured by C_1. Then $x_{0,1} \in \{0, 2\}$. Assume $x_{0,1} = 0$. Then, either $x_{1,1} = x_{2,1} = 0$ or $x_{1,1} = 1$, $x_{2,1} = 2$. In the former case neither $\mathbf{x_1}$ nor $\mathbf{x_2}$ is coloured by C_3, which contradicts $\mathbf{x_0}$, $\mathbf{x_1}$, $\mathbf{x_2}$ being rainbow. In the latter case, suppose that $\mathbf{x_1}$ is coloured by C_2. Then $\mathbf{x_2}$ is coloured by C_3. Hence, $\mathbf{x_1} \in \{101, 111\}$ and $\mathbf{x_2} \in \{212, 211\}$. It follows that either $\mathbf{x_1} = 111$ and $\mathbf{x_2} = 211$ or $\mathbf{x_1} = 111$ and $\mathbf{x_2} = 212$. Then $\mathbf{x_0} = 011$ or $\mathbf{x_0} = 010$. This contradicts the assumption that $\mathbf{x_0}$ is coloured by C_1. Other cases are handled similarly.

Another generic direction we considered is increasing the number of colours and the length of a rainbow AP.

Proposition 5.1. *For every n and $k > 3$, there exists a k-colouring of $[n]$ with no rainbow $AP(k)$ and with each colour of size at least $\lfloor \frac{n+2}{3\lfloor (k+4)/3 \rfloor} \rfloor$.*

Proof. First, we partition the set of k colours into three sets C_1, C_2, and C_3 of sizes l_1, l_2, and l_3, respectively, where (l_1, l_2, l_3) is defined as follows:

$$(l_1, l_2, l_3) = \begin{cases} (l+1, l, l-1) & \text{if } k = 3l, \\ (l+1, l+1, l-1) & \text{if } k = 3l+1, \\ (l+2, l, l) & \text{if } k = 3l+2. \end{cases}$$

Notice that by the above definition, $\max(l_1, l_2, l_3) = \lfloor (k+4)/3 \rfloor$, and there are always $i, j \in \{1, 2, 3\}$ such that $|l_i - l_j| = 2$. Now, for $i = 1, 2, 3$, we colour the numbers in

$N_i := \{x \in [n] : x \equiv i \pmod 3\}$ with colours in C_i, so that for each two colours in C_i, the number of times they are used differ by at most 1. Thus, it is easy to verify that each colour is used at least $\lfloor \frac{n+2}{3\max(l_1,l_2,l_3)} \rfloor = \lfloor \frac{n+2}{3\lceil (k+4)/3 \rceil} \rfloor$ times. Also, every arithmetic progression $\mathscr{A}$ is either completely contained in one of the N_is, or satisfies $\big||\mathscr{A} \cap N_i| - |\mathscr{A} \cap N_j|\big| \leqslant 1$ for every $i, j \in \{1, 2, 3\}$. Thus, the existence of i, j with $|l_i - l_j| = 2$ shows that there is no rainbow $AP(k)$ in this colouring. □

The above proposition can be thought of as a generalization of Proposition 2.11 for $k > 3$. One is tempted to also generalize Theorem 1.3 and conjecture that any partition $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_k$ into k colour classes, with every colour class having density greater than $\frac{1}{3\lceil (k+4)/3 \rceil}$, contains a rainbow $AP(k)$. However, it is easy to verify that the following equinumerous colourings of $\mathbb{N}$ do not contain any rainbow $AP(5)$, and hence the generalization of Radoičić's conjecture is not true for $k = 5, 6$. Namely, *e.g.*, for c_5, it suffices to check that no $AP(5)$ in [10] is rainbow, that is,

$$c_5(i) := \begin{cases} 1 & \text{if } i \equiv 1,3 \pmod{10}, \\ 2 & \text{if } i \equiv 2,5 \pmod{10}, \\ 3 & \text{if } i \equiv 4,8 \pmod{10}, \\ 4 & \text{if } i \equiv 6,7 \pmod{10}, \\ 5 & \text{if } i \equiv 9,0 \pmod{10}, \end{cases} \qquad c_6(i) := \begin{cases} 1 & \text{if } i \equiv 1,3 \pmod{12}, \\ 2 & \text{if } i \equiv 2,4 \pmod{12}, \\ 3 & \text{if } i \equiv 5,7 \pmod{12}, \\ 4 & \text{if } i \equiv 6,8 \pmod{12}, \\ 5 & \text{if } i \equiv 9,11 \pmod{12}, \\ 6 & \text{if } i \equiv 10,0 \pmod{12}. \end{cases}$$

We still do not know whether there is a similar example when the number of colours is $k = 4$ or $k > 6$. If the number of colours is infinite, the following proposition shows that one cannot guarantee even the existence of a rainbow $AP(3)$ with the assumption that each colour has a positive density.

Proposition 5.2. *There is a colouring of $\mathbb{N}$ with infinitely many colours, with each colour having positive density such that there is no rainbow $AP(3)$.*

Proof. For each $x \in \mathbb{N}$, let $c(x)$ be the largest integer k such that x is divisible by 3^k. It is easy to see that the colour k has density $2{\cdot}3^{-k-1} > 0$ in this colouring. Also, if $c(x) \neq c(y)$, it is not difficult to see that $c(2y - x) = c(2x - y) = c((x + y)/2) = \min(c(x), c(y))$. Therefore, if two elements of an arithmetic progression are coloured with two different colours, the third term must be coloured with one of those two colours. Thus, there is no rainbow $AP(3)$ in c. □

Yet another direction is limiting our attention to equinumerous colourings and letting the number of colours be different from the desired length of a rainbow AP. Let T_k denote the minimal number $t \in \mathbb{N}$ such that there is a rainbow $AP(k)$ in every equinumerous t-colouring of $[tn]$ for every $n \in \mathbb{N}$. We have the following lower and upper bounds on T_k.

Proposition 5.3. *For every $k \geqslant 3$, $\lfloor \frac{k^2}{4} \rfloor < T_k \leqslant \frac{k(k-1)^2}{2}$.*

Proof. First, we prove the upper bound. Let $m = a(k-1) + b$, with $k \geqslant 3$, $a \geqslant 1$, and $0 \leqslant b \leqslant k-1$. We note that there is bijective correspondence between the set of all $AP(k)$s and the set of all 2-element sets $\{\alpha, \beta\} \subseteq [m]$, $\alpha < \beta$, with $\alpha \equiv \beta \pmod{k-1}$. It follows that the number of all $AP(k)$s in $[m]$ is $b\binom{a+1}{2} + (k-b-1)\binom{a+1}{2}$. Thus,

$$\#\text{ of } AP(k)\text{s in } [tn] > \frac{tn(tn - 2(k-1))}{2(k-1)}.$$

Note that for a t-regular colouring of $[tn]$, in each of the t colours there are $\binom{n}{2}$ pairs of numbers. The numbers from each of these pairs can be the terms of at most $\binom{k}{2}$ different $AP(k)$s. Therefore, for any t-regular colouring of $[tn]$ there are at most $t\binom{k}{2}\binom{n}{2}$ $AP(k)$s that are *not* rainbow. Therefore, T_k is bounded by the smallest t that satisfies

$$\frac{tn(tn - 2(k-1))}{2(k-1)} \geqslant t\binom{k}{2}\binom{n}{2} \text{ for all } n,$$

which implies the upper bound.

As for the lower bound, we will exhibit colourings c_1 and c_2, showing that $T_{2k+1} > k^2 + k$ and $T_{2k} > k^2$.

Let a *j-block* B_j ($j \in \mathbb{N}$) be the sequence $12\ldots j12\ldots j$, where the *left half* and the *right half* of the block are naturally defined. For $a \in \mathbb{Z}$, let $B_j + a$ be the sequence $(a+1)(a+2)\ldots(a+j)(a+1)(a+2)\ldots(a+j)$.

We define the colouring c_1 of $[2k^2 + 2k]$ in the following way (bars denoting endpoints of the blocks):

$$|B_k^-|\ldots|B_j^-|\ldots|B_2^-|B_1^-|B_1^+|B_2^+|\ldots|B_i^+|\ldots|B_k^+|,$$

where $B_j^- = B_j - \binom{j+1}{2}$ and $B_i^+ = B_i + \binom{i}{2}$. Note that c_1 uses each of the $k^2 + k$ colours exactly twice.

The colouring c_2 of $[2k^2]$ is defined similarly:

$$|B_{k-1}^-|\ldots|B_j^-|\ldots|B_2^-|B_1^-|B_1^+|B_2^+|\ldots|B_i^+|\ldots|B_k^+|,$$

thus using each of the k^2 colours exactly twice.

Next, we show that $[2k^2 + 2k]$, coloured by c_1, does not contain a rainbow $AP(2k+1)$. The key observation is that a rainbow AP with common difference d cannot contain elements from opposite halves of any block B_j, where d divides j. Fix a longest rainbow AP $\mathscr{A}$ and let d denote its common difference. If $d > k$, then the length of $\mathscr{A}$ is $\leqslant 2k$. If $d \leqslant k$, then $\mathscr{A}$ is one of the following three types.

(1) $\mathscr{A}$ is contained in $|B_d^-|B_{d-1}^-|\ldots|B_2^-|B_1^-|B_1^+|B_2^+|\ldots|B_{d-1}^+|B_d^+|$.
Then $\mathscr{A}$ does not intersect either the left half of B_d^- or the right half of B_d^+. Hence, the length of $\mathscr{A}$ is at most $2d \leqslant 2k$.

(2) $\mathscr{A}$ is contained in $|B_{(j+1)d}^-|B_{(j+1)d-1}^-|\ldots|B_{jd}^-|$ or in $|B_{jd}^+|B_{jd+1}^+|\ldots|B_{(j+1)d}^+|$, where $(j+1)d \leqslant k$.
Assume that the first case occurs. Then $\mathscr{A}$ does not intersect either the left half of $B_{(j+1)d}^-$ or the right half of B_{jd}^-. Hence, the length of $\mathscr{A}$ is at most

$$\frac{1}{d}(jd + 2(jd+1) + 2(jd+2) + \cdots + 2(jd+d-1) + (jd+d) \leqslant 2(j+1)d \leqslant 2k.$$

(3) $\mathscr{A}$ is contained in $|B^-_{jd+x}|B^-_{jd+x-1}|\ldots|B^-_{jd}|$ or in $|B^+_{jd}|B^+_{jd+1}|\ldots|B^+_{jd+x}|$, where $jd+x<k$. Assume that the first case occurs. Then $\mathscr{A}$ does not intersect the right half of B^-_{jd}. Hence, the length of $\mathscr{A}$ is at most

$$\begin{aligned}&\frac{1}{d}(jd+2(jd+1)+2(jd+2)+\cdots+2(jd+x-1)+2(jd+x))\\&\quad\leqslant\frac{1}{d}(jd+2jd(d-1)+d(x-1))<2(jd+x)<2k.\end{aligned}$$

Similarly, we show that $[2k^2]$, coloured by c_2, does not contain a rainbow $AP(2k)$. □

Note that Proposition 5.3 gives $3\leqslant T_3\leqslant 6$, while Conjecture 1.1 claims that $T_3=3$.

Conjecture 5.4. *For all $k\geqslant 3$, $T_k=\Theta(k^2)$.*

The proof of Proposition 5.3 above is inspired by the proof of the following 'canonical version' of van der Waerden's theorem on arithmetic progressions, due to Erdős and Graham [8]. We include this for completeness.

Theorem 5.5. *For every positive integer $k\geqslant 3$, there exists a positive integer $n(k)$ such that every colouring of the first $n\geqslant n(k)$ positive integers contains either a monochromatic $AP(k)$ or a rainbow $AP(k)$.*

Proof. By Szemerédi's theorem [32], for every $\delta>0$ there exists a positive integer $s(k,\delta)$ such that for all $n\geqslant s(k,\delta)$ every subset $C\subset[n]$ with $|C|>\delta n$ contains an $AP(k)$. Fix $\delta=\frac{2}{k(k-1)^2}$ and let $n\geqslant s(k,\delta)$. Suppose there exists a colouring of $[n]=C_1\cup C_2\cup\cdots\cup C_r$ containing no monochromatic or rainbow $AP(k)$. Since a colour class C_i does not contain a monochromatic $AP(k)$, then $|C_i|\leqslant\delta n$. In the proof of Proposition 5.3, it was shown that the number of $AP(k)$s in $[n]$ is at least $\frac{n(n-2(k-1))}{2(k-1)}$. Since every non-rainbow $AP(k)$ contains a pair of terms of the same colour, there are at most $\binom{k}{2}\sum_{i=1}^{r}\binom{|C_i|}{2}$ non-rainbow $AP(k)$s. Since $0\leqslant|C_i|\leqslant\delta n$ and $\sum_{i=1}|C_i|=n$, we have the inequality

$$\binom{k}{2}\sum_{i=1}^{r}\binom{|C_i|}{2}\leqslant\binom{k}{2}\sum_{i=1}^{1/\delta}\binom{\delta n}{2}=\frac{\binom{k}{2}\binom{\delta n}{2}}{\delta}.$$

However, the inequality $\frac{n(n-2(k-1))}{2(k-1)}\leqslant\frac{\binom{k}{2}\binom{\delta n}{2}}{\delta}$ does not hold for our choice of δ. □

It is easy to show that the maximal number of rainbow $AP(3)$s over all equinumerous 3-colourings of $[3n]$ is $\lfloor 3n^2/2\rfloor$, this being achieved for the unique 3-colouring with colour classes $R=\{n|n\equiv 0 \pmod 3\}$, $B=\{n|n\equiv 1 \pmod 3\}$ and $G=\{n|n\equiv 2 \pmod 3\}$. It seems very difficult to characterize those equinumerous 3-colourings (in general, k-colourings) that minimize the number of rainbow $AP(3)$s. Letting $f_k(n)$ denote the minimal number of rainbow $AP(k)$s, over all equinumerous k-colourings of $[kn]$, we pose the following conjecture.

Conjecture 5.6. $f_3(n)=\Omega(n)$.

If we define $g_k(n)$ as the minimal number of rainbow $AP(k)$s, over all equinumerous k-colourings of $\mathbb{Z}_{kn}$, then a straightforward counting argument shows that $g_3(n) \geqslant n$, when n is odd.

Finally, the further generalization of Vosper's theorem, due to Serra and Zémor [30], may lead to a generalization of Theorem 4.1 for more than 3 colour classes.

Acknowledgements

Parts of this research were done while the second author was working at the RSI program at MIT. We would like to thank the Center for Excellence in Education and Professor Hartley Rogers for their support of this program. The first and the fifth authors did parts of this research while visiting DIMATIA at the Charles University and would like to thank Jan Kratochvíl, Jirka Matoušek and Pavel Valtr for their hospitality.

References

[1] Alekseev, V. E. and Savchev, S. (1987) Problem M. 1040. *Kvant* **4** 23.

[2] Alon, N. (1983) On a conjecture of Erdős, Simonovits and Sós concerning anti-Ramsey numbers. *J. Graph Theory* **7** 91–94.

[3] Alon, N., Lefmann, H. and Rödl, V. (1991) On an anti-Ramsey type result. In *Sets, Graphs and Numbers*, Budapest (Hungary), Vol. 60 of *Coll. Math. Soc. J. Bolyai*, pp. 9–22.

[4] Alon, N., Jiang, T., Miller, Z. and Pritikin, D. Properly colored subgraphs and rainbow subgraphs in edge-coloring with local constraints. To appear in *Random Struct. Alg.*

[5] Axenovich, M., Füredi, Z. and Mubayi, D. (2000) On generalized Ramsey theory: The bipartite case. *J. Combin. Theory Ser. B* **79** 66–86.

[6] Chung, F. R. K. and Graham, R. L. (1975) On multicolor Ramsey numbers for bipartite graphs. *J. Combin. Theory Ser. B* **18** 164–169.

[7] Deuber, W., Graham, R. L., Prömel, H. J. and Voigt, B. (1983) A canonical partition theorem for equivalence relations in $\mathbb{Z}^t$. *J. Combin. Theory Ser. A* **34** 331–339.

[8] Erdős, P. and Graham, R. L. (1980) *Old and New Problems and Results in Combinatorial Number Theory*, Vol. 28 of *L'Enseignement Mathématique*, Université de Genève.

[9] Erdős, P. and Rado, R. (1950) A combinatorial theorem. *J. London Math. Soc.* **25** 249–255.

[10] Erdős, P., Simonovits, M. and Sós, V. T. (1973) Anti-Ramsey theorems. In *Coll. Math. Soc. J. Bolyai*, Vol. 10 of *Infinite and Finite Sets*, Keszthely (Hungary), pp. 633–642.

[11] Erdős, P. and Tuza, Zs. (1993) Rainbow subgraphs in edge-colorings of complete graphs. In *Quo Vadis, Graph Theory?* (J. Gimbel, J. W. Kennedy and L. V. Quintas, eds), Vol. 55 of *Ann. Discrete Math.*, pp. 81–88.

[12] Freiman, G. A. (1961) Inverse problems of additive number theory: On the addition of sets of residues with respect to a prime modulus. *Soviet. Math. Doklady* **2** 1520–1522.

[13] Gowers, W. T. (2001) A new proof of Szemerédi's theorem. *Geom. Funct. Anal.* **11** 465–588.

[14] Graham, R. L., Rothschild, B. L. and Spencer, J. H. (1990) *Ramsey Theory*, Wiley.

[15] Guy, R. K. (1994) *Unsolved Problems in Number Theory*, Springer.

[16] Hamidoune, Y. O. and Rødseth, Ø. J. (2000) An inverse theorem mod *p*. *Acta Arithmetica* **92** 251–262.

[17] Jamison, R. E., Jiang, T. and Ling, A. C. (2003) Constrained Ramsey numbers of graphs. *J. Graph Theory* **42** 1–16.

[18] Lefmann, H. (1986) A canonical version for partition regular systems of linear equations. *J. Combin. Theory Ser. A* **41** 95–104.

[19] Lefmann, H. and Rödl, V. (1993) On canonical Ramsey numbers for complete graphs versus paths. *J. Combin. Theory Ser. B* **58** 1–13.
[20] Lefmann, H. and Rödl, V. (1995) On Erdős–Rado numbers. *Combinatorica* **15** 85–104.
[21] Montágh, B. Anti-Ramsey theorems on spanning trees. Submitted.
[22] Montellano-Ballesteros, J. J. and Neumann-Lara, V. (2002) An anti-Ramsey theorem. *Combinatorica* **22** 445–449.
[23] Nathanson, M. B. (1996) *Additive Number Theory: Inverse Problems and the Geometry of Sumsets*, Vol. GTM 165, Springer-Verlag.
[24] Nešetřil, J., Prömel, H. J., Rödl, V. and Voigt, B. (1985) Canonizing ordering theorems for Hales–Jewett structures. *J. Combin. Theory Ser. A* **40** 394–408.
[25] Prömel, H. J. and Voigt, B. (1983) Canonical partition theorems for parameter sets. *J. Combin. Theory Ser. A* **35** 309–327.
[26] Richer, D. (2000) Unordered canonical Ramsey numbers. *J. Combin. Theory Ser. B* **80** 172–177.
[27] Rödl, V. and Tuza, Zs. (1992) Rainbow subgraphs in properly edge-colored graphs. *Random Struct. Alg.* **3** 175–182.
[28] Schönheim, J. (1990) On partitions of the positive integers with no x, y, z belonging to distinct classes satisfying $x + y = z$. In *Number Theory: Proceedings of the First Conference of the Canadian Number Theory Association, Banff 1988* (R. A. Mollin, ed.), de Gruyter, pp. 515–528.
[29] Schur, I. (1916) Über die Kongruenz $x^m + y^m \equiv z^m \bmod p$. *Jahresb. Deutsche Math. Verein* **25** 114–117.
[30] Serra, O. and Zémor, G. (2000) On a generalization of a theorem by Vosper. *Integers. Electron. J. Combin. Number Theory* **0** A10 (electronic, `http://www.integers-ejcnt.org/`).
[31] Székely, G., ed. (1995) *Contests in Higher Mathematics, Miklós Schweitzer Competition 1962–1991*, Problem Books in Mathematics, Springer.
[32] Szemerédi, E. (1975) On sets of integers containing no k elements in arithmetic progression. *Acta Arithmetica* **27** 199–245.
[33] Vosper, G. (1956) The critical pairs of subsets of a group of prime order. *J. London Math. Soc.* **31** 200–205.