

# Ramsey-type problem for an almost monochromatic $K_4$

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## Abstract

In this short note we prove that there is a constant  $c$  such that every  $k$ -edge-coloring of the complete graph  $K_n$  with  $n \geq 2^{ck}$  contains a  $K_4$  whose edges receive at most two colors. This improves on a result of Kostochka and Mubayi, and is the first exponential bound for this problem.

## 1 Introduction

The *Ramsey number*  $R(t; k)$  is the least positive integer  $n$  such that every  $k$ -coloring of the edges of the complete graph  $K_n$  contains a monochromatic  $K_t$ . Schur in 1916 showed that  $R(3; k)$  is at least exponential in  $k$  and at most a constant times  $k!$ . Despite various efforts over the past century to determine the asymptotics of  $R(t; k)$ , there were only improvements in the exponential constant in the lower bound and the constant factor in the upper bound. It is a major open problem to determine whether or not there is a constant  $c$  such that  $R(3; k) \leq c^k$  for all  $k$  (see, e.g., the monograph [9]).

In 1981, Erdős [6] proposed to study the following generalization of the classical Ramsey problem. Let  $p, q$  be positive integers with  $2 \leq q \leq \binom{p}{2}$ . A  $(p, q)$ -coloring of  $K_n$  is an edge-coloring such that every copy of  $K_p$  receives at least  $q$  distinct colors. Let  $f(n, p, q)$  be the minimum number of colors in a  $(p, q)$ -coloring of  $K_n$ . Determining the numbers  $f(n, p, 2)$  is equivalent to determining the multicolor Ramsey numbers  $R(p; k)$  as an edge-coloring is a  $(p, 2)$ -coloring if and only if it does not contain a monochromatic  $K_p$ . Over the last two decades, the study of  $f(n, p, q)$  drew a lot of attention. Erdős and Gyárfás [7] proved several results on  $f(n, p, q)$ , e.g., they determined for which fixed  $p$  and  $q$  we have  $f(n, p, q)$  is at least linear in  $n$ , quadratic in  $n$ , or  $\binom{n}{2}$  minus a constant. For fixed  $p$ , they also gave bounds on the smallest  $q$  for which  $f(n, p, q)$  is asymptotically  $\binom{n}{2}$ . These bounds were significantly tightened by Sárközy and Selkow [15] using Szemerédi's Regularity Lemma. In a different paper, Sárközy and Selkow [14] show that  $f(n, p, q)$  is linear in  $n$  for at most  $\log p$  values of  $q$ . (Here, and throughout the paper, all logarithms are base 2.) There are also results on the behavior of  $f(n, p, q)$  for particular values of  $p$  and  $q$ . Mubayi [13] gave an explicit construction of an edge-coloring which together with the already known lower bound shows that  $f(n, 4, 4) = n^{1/2+o(1)}$ . Using Behrend's construction of a dense set with no arithmetic progressions of length three, Axenovich [2] showed that

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$\frac{1+\sqrt{5}}{2}n - 3 \leq f(n, 5, 9) \leq 2n^{1+c/\sqrt{\log n}}$ . These examples demonstrate that special cases of  $f(n, p, q)$  lead to many interesting problems.

As was pointed out by Erdős and Gyárfás [7], one of the most intriguing problems among the small cases is the behavior of  $f(n, 4, 3)$ . This problem can be rephrased in terms of another more convenient function. Let  $g(k)$  be the largest positive integer  $n$  for which there is a  $k$ -edge-coloring of  $K_n$  in which every  $K_4$  receives at least three colors, i.e., for which  $f(n, 4, 3) \leq k$ . Restated,  $g(k) + 1$  is the smallest positive integer  $n$  for which every  $k$ -edge-coloring of the edges of  $K_n$  contains a  $K_4$  that receives at most two colors. In 1981, by an easy application of the probabilistic method, Erdős [6] showed that  $g(k)$  is superlinear in  $k$ . Later, Erdős and Gyárfás used the Lovász Local Lemma to show that  $g(k)$  is at least quadratic in  $k$ . Mubayi [12] improved these bounds substantially, showing that  $g(k) \geq 2^{c(\log k)^2}$  for some absolute positive constant  $c$ . On the other hand, the progress on the upper bound was much slower. Until very recently, the best result was of the form  $g(k) < k^{ck}$  for some constant  $c$ , which follows trivially from the multicolor  $k$ -color Ramsey number for  $K_4$ . This bound was improved by Kostochka and Mubayi [10], who showed that  $g(k) < (\log k)^{ck}$  for some constant  $c$ . Here we further extend their neat approach and obtain the first exponential upper bound for this problem.

**Theorem 1.1** *For  $k > 2^{100}$ , we have  $g(k) < 2^{2000k}$ .*

While it is a longstanding open problem to determine whether or not  $R(t; k)$  grows faster than exponential in  $k$ , it is not difficult to prove an exponential upper bound if we restrict the colorings to those that do not contain a rainbow  $K_s$  for fixed  $s$ . Let  $M(k, t, s)$  be the minimum  $n$  such that every  $k$ -edge-coloring of  $K_n$  has a monochromatic  $K_t$  or a rainbow  $K_s$ . Axenovich and Iverson [4] showed that  $M(k, t, 3) \leq 2^{kt^2}$ . We improve on their bound by showing that  $M(k, t, s) \leq s^{4kt}$  for all  $k, t, s$ . In the other direction, we prove that for all positive integers  $k$  and  $t$  with  $k$  even and  $t \geq 3$ ,  $M(k, t, 3) \geq 2^{kt/4}$ , thus determining  $M(k, t, 3)$  up to a constant factor in the exponent.

The rest of this paper is organized as follows. In the next section, we prove our main result, Theorem 1.1. In Section 3, we study the Ramsey problem for colorings without rainbow  $K_s$ . The last section of this note contains some concluding remarks. Throughout the paper, we systematically omit floor and ceiling signs whenever they are not crucial for the sake of clarity of presentation. We also do not make any serious attempt to optimize absolute constants in our statements and proofs.

## 2 Proof of Theorem 1.1

Our proof develops further on ideas in [10]. Like the Kostochka-Mubayi proof, we show that the  $K_4$  we find is monochromatic or is a  $C_4$  in one color and a matching in the other color. Call a coloring of  $K_t$  *rainbow* if all  $\binom{t}{2}$  edges have different colors. Let  $g(k, t)$  be the largest positive integer  $n$  such that there is a  $k$ -edge-coloring of  $K_n$  with no rainbow  $K_t$ , and in which the edges of every  $K_4$  have at least three colors. We will study  $g(k)$  by investigating the behavior of  $g(k, t)$ .

Before jumping into the details of the proof of Theorem 1.1, we first outline the proof idea. Note that  $g(k) = g(k, k)$  for  $k > 2$  as a rainbow  $K_k$  would use  $\binom{k}{2} > k$  colors. We give a recursive upper bound on  $g(k, t)$  which implies Theorem 1.1. We first prove a couple of lemmas which show that in any

edge-coloring without a rainbow  $K_t$ , there are many vertices that have large degree in some color  $i$ . We then apply a simple probabilistic lemma to find a large subset  $V_2$  of vertices such that every vertex subset of size  $d$  (with  $d \ll t$ ) has many common neighbors in color  $i$ . We use this to get an upper bound on  $g(k, t)$  as follows. Consider a  $k$ -edge-coloring of  $K_n$  with  $n = g(k, t)$  without a rainbow  $K_t$  and with every  $K_4$  containing at least three colors. There are two possible cases. If there is no rainbow  $K_d$  in the set  $V_2$ , then we obtain an upper bound on  $g(k, t)$  using the fact that  $|V_2|$  has size at most  $g(k, d)$ . If there is a set  $R \subset V_2$  of  $d$  vertices which forms a rainbow  $K_d$ , then the  $\binom{d}{2}$  colors that appear in this rainbow  $K_d$  cannot appear in the edges inside the set  $N_i(R)$  of vertices that are adjacent to every vertex in  $R$  in color  $i$ , for otherwise we would obtain a  $K_4$  having at most two colors (the color  $i$  and the color that appears in both  $R$  and in  $N_i(R)$ ). In this case we obtain an upper bound on  $g(k, t)$  using the fact that  $|N_i(R)| \leq g(k - \binom{d}{2}, t)$ . Finally, if the coloring has no rainbow  $K_d$  with  $d$  constant, it is easy to show an exponential upper bound.

For an edge-coloring of  $K_n$ , a vertex  $x$ , and a color  $i$ , let  $d_i(x)$  denote the degree of vertex  $x$  in color  $i$ . Our first lemma shows that if, for every vertex  $x$  and color  $i$ ,  $d_i(x)$  is not too large, then the coloring contains many rainbow cliques.

**Lemma 2.1** *If an edge-coloring of the complete graph  $K_n$  satisfies  $d_i(x) \leq \delta n$  for each  $x \in V(K_n)$  and each color  $i$ , then this coloring has at most  $\frac{5}{8}\delta t^4 \binom{n}{t}$  non-rainbow copies of  $K_t$ .*

**Proof.** If a  $K_t$  is not rainbow, then it has two adjacent edges of the same color or two nonadjacent edges of the same color. We will use this fact to give an upper bound on the number of  $K_t$ s that are not rainbow.

Let  $\nu(i, t, n)$  be the number of copies of  $K_t$  in  $K_n$  in which there is a vertex in at least two edges of color  $i$ . We can first choose the vertex, and then the two edges with color  $i$ . Hence, the number of  $K_t$ s for which there is a vertex with degree at least two in some color is at most

$$\begin{aligned} \sum_i \nu(i, t, n) &\leq \sum_i \sum_{x \in V} \binom{d_i(x)}{2} \binom{n-3}{t-3} \leq n\delta^{-1} \binom{\delta n}{2} \binom{n-3}{t-3} \\ &\leq \frac{\delta n^3}{2} \left(\frac{t}{n}\right)^3 \binom{n}{t} = \frac{1}{2}\delta t^3 \binom{n}{t}, \end{aligned}$$

where we used that  $d_i(x) \leq \delta n$  together with the convexity of the function  $f(y) = \binom{y}{2}$ .

Let  $\psi(i, t, n)$  be the number of copies of  $K_t$  in  $K_n$  in which there is a matching of size at least two in color  $i$ . Let  $e_i$  denote the number of edges of color  $i$ . Since

$$e_i \leq \frac{n}{2} \max_{x \in V} d_i(x) \leq \frac{\delta}{2} n^2,$$

then the number of  $K_t$ s in which there is a matching of size at least two in some color is at most

$$\sum_i \psi(i, t, n) \leq \sum_i \binom{e_i}{2} \binom{n-4}{t-4} \leq \delta^{-1} \binom{\delta n^2/2}{2} \binom{n-4}{t-4} \leq \frac{\delta t^4}{8} \binom{n}{t},$$

where again we used the convexity of the function  $f(y) = \binom{y}{2}$ . Hence, the number of  $K_t$ s which are not rainbow is at most  $\frac{1}{2}\delta t^3 \binom{n}{t} + \frac{1}{8}\delta t^4 \binom{n}{t} \leq \frac{5}{8}\delta t^4 \binom{n}{t}$ , completing the proof.  $\square$

For the proof of Theorem 1.1, we do not need the full strength of this lemma since we will only use the existence of at least one rainbow  $K_t$ . We also would like to mention the following stronger result. Call an edge-coloring  $m$ -good if each color appears at most  $m$  times at each vertex. Let  $h(m, t)$  denote the minimum  $n$  such that every  $m$ -good edge-coloring of  $K_n$  contains a rainbow  $K_t$ . The above lemma demonstrates that  $h(m, t)$  is at most  $mt^4$ . It is shown by Alon, Jiang, Miller, and Pritikin [1] that there are constant positive constants  $c_1$  and  $c_2$  such that

$$c_1 mt^3 / \log t \leq h(m, t) \leq c_2 mt^3 / \log t.$$

The following easy corollary of Lemma 2.1 demonstrates that in every  $k$ -edge-coloring without a rainbow  $K_t$ , there is a color and a large set of vertices which have large degree in that color.

**Corollary 2.2** *In every  $k$ -edge-coloring of  $K_n$  without a rainbow  $K_t$ , there is a subset  $V_1 \subset V(K_n)$  with  $|V_1| \geq \frac{n}{2k}$  and a color  $i$  such that  $d_i(x) \geq \frac{n}{2t^4}$  for each vertex  $x \in V_1$ .*

**Proof.** Let  $V' \subset V(K_n)$  be those vertices  $x$  for which there is a color  $i$  such that  $d_i(x) \geq \frac{n}{2t^4}$ .

Case 1:  $|V'| < n/2$ . In this case, letting  $V'' = V(K_n) \setminus V'$ ,  $|V''| \geq n/2$  and no vertex in  $V''$  has degree at least  $\frac{n}{2t^4} \leq |V''|/t^4$  in any given color. By Lemma 2.1 applied to the coloring of  $K_n$  restricted to  $V''$  with  $\delta = t^{-4}$ , there are at least  $\frac{3}{8} \binom{|V''|}{t}$  rainbow  $K_t$ s, contradicting the assumption that the coloring is free of rainbow  $K_t$ s.

Case 2:  $|V'| \geq n/2$ . In this case, by the pigeonhole principle, there is a color  $i$  and at least  $\frac{n}{2k}$  vertices  $x$  for which  $d_i(x) \geq \frac{n}{2t^4}$ , completing the proof.  $\square$

The following lemma is essentially the same as results in [11] and [16]. Its proof uses a probabilistic argument commonly referred to as dependent random choice, which appears to be a powerful tool in proving various results in Ramsey theory (see, e.g., [8] and its references). In a graph  $G$ , the *neighborhood*  $N(v)$  of a vertex  $v$  is the set of vertices adjacent to  $v$ . For a vertex subset  $U$  of a graph  $G$ , the *common neighborhood*  $N(U)$  is the set of vertices adjacent to all vertices in  $U$ .

**Lemma 2.3** *Let  $G = (V, E)$  be a graph with  $n$  vertices and  $V_1 \subset V$  be a subset with  $|V_1| = m$  in which each vertex has degree at least  $\alpha n$ . If  $\beta \leq m^{-d/h}$ , then there is a subset  $V_2 \subset V_1$  with  $|V_2| \geq \alpha^h m - 1$  such that every  $d$ -tuple in  $V_2$  has at least  $\beta n$  common neighbors.*

**Proof.** Let  $U = \{x_1, \dots, x_h\}$  be a subset of  $h$  random vertices from  $V$  chosen uniformly with repetitions, and let  $V'_1 = N(U) \cap V_1$ . We have

$$\mathbb{E}[|V'_1|] = \sum_{v \in V_1} \Pr(v \in N(U)) = \sum_{v \in V_1} \left( \frac{|N(v)|}{n} \right)^h \geq \alpha^h m.$$

The probability that a given set  $W \subset V_1$  of vertices is contained in  $V'_1$  is  $\left( \frac{|N(W)|}{n} \right)^h$ . Let  $Z$  denote the number of  $d$ -tuples in  $V'_1$  with less than  $\beta n$  common neighbors. So

$$\mathbb{E}[Z] = \sum_{W \subset V_1, |W|=d, |N(W)| < \beta n} \Pr(W \subset V_2) \leq \binom{m}{d} \beta^h \leq m^d \beta^h \leq 1.$$

Hence, the expectation of  $|V'_1| - Z$  is at least  $\alpha^h m - 1$  and thus, there is a choice  $U_0$  for  $U$  such that the corresponding value of  $|V'_1| - Z$  is at least  $\alpha^h m - 1$ . For every  $d$ -tuple  $D$  of vertices of  $V'_1$  with less than  $\beta n$  common neighbors, delete a vertex  $v_D \in D$  from  $V'_1$ . Letting  $V_2$  be the resulting set, it is clear that  $V_2$  has the desired properties, completing the proof.  $\square$

The proof of the next lemma uses the standard pigeonhole argument together with Lemma 2.1.

**Lemma 2.4** *Let  $d, k$  be integers with  $d, k \geq 2$ . Then every  $k$ -edge-coloring of  $K_n$  with  $n \geq d^{12k}$  and without a rainbow  $K_d$  has a monochromatic  $K_4$ . In particular, we have  $g(k, d) < d^{12k}$ .*

**Proof.** Suppose for contradiction that there is a  $k$ -edge-coloring of  $K_n$  with  $n \geq d^{12k}$  and without a rainbow  $K_d$  and without a monochromatic  $K_4$ . By Lemma 2.1 with  $t = d$  and  $\delta = d^{-4}$ , this graph contains a vertex  $x_1$  with degree at least  $\frac{n}{d^4}$  in some color  $c_1$ . Pick this vertex  $x_1$  out and let  $N_1$  be the set of vertices adjacent to  $x_1$  by color  $c_1$ . We will define a sequence  $x_1, \dots, x_{2k+1}$  of vertices, a sequence  $c_1, \dots, c_{2k+1}$  of colors, and a sequence  $V(K_n) \supset N_1 \supset \dots \supset N_{2k+1}$  of vertex subsets. Once  $x_j, c_j$ , and  $N_j$  have been defined, pick a vertex  $x_{j+1}$  in  $N_j$  in at least  $\frac{|N_j|}{d^4}$  edges in some color  $c_{j+1}$  with other vertices in  $N_j$ . Pick this vertex  $x_{j+1}$  out and let  $N_{j+1}$  be the set of vertices in  $N_j$  that are adjacent to  $x_j$  by color  $c_j$ . Note that  $|N_{j+1}| \geq d^{-4}|N_j|$  so

$$|N_{2k+1}| \geq (d^{-4})^{2k+1} n \geq 1.$$

Therefore, there is a color  $c$  such that  $c$  is represented at least three times in the list  $c_1, \dots, c_{2k+1}$  and the three vertices  $x_{j_1}, x_{j_2}, x_{j_3}$  together with a vertex from  $N_{2k+1}$  form a monochromatic  $K_4$  in color  $c$ , where  $c_{j_1} = c_{j_2} = c_{j_3} = c$  with  $j_1 < j_2 < j_3$ .  $\square$

**Lemma 2.5** *Let  $d, k, t$  be positive integers with  $3 \leq d \leq t$  and  $d \geq 40 \log t$ . If  $k \geq \binom{d}{2}$ , then*

$$g(k, t) \leq \max \left( 4kg(k, t)^{\frac{20 \log t}{d}} g(k, d), 2^{\binom{d}{2}} g \left( k - \binom{d}{2}, t \right) \right). \quad (1)$$

*Otherwise, we have  $g(k, t) = g(k, d)$ .*

**Proof.** Note that if  $k < \binom{d}{2}$ , then a  $k$ -edge-coloring cannot have a rainbow  $K_d$ . Therefore,  $g(k, t) = g(k, d)$  in this case. So we assume  $k \geq \binom{d}{2}$ . By the definition of  $g(k, t)$ , there is a  $k$ -edge-coloring of  $K_n$  with  $n = g(k, t)$  with no rainbow  $K_t$  and in which every  $K_4$  receives at least three colors. Consider such a coloring. By Corollary 2.2, there is a color  $i$  and a subset  $V_1 \subset V(K_n)$  with  $|V_1| \geq \frac{n}{2k}$  and  $d_i(x) \geq \frac{n}{2t^4}$  for every vertex  $x \in V_1$ . Apply Lemma 2.3 to the graph of color  $i$  with  $\alpha = \frac{1}{2t^4}$ ,  $\beta = 2^{-\binom{d}{2}}$ ,  $m = |V_1| \geq \frac{n}{2k}$ , and  $h = 4d^{-1} \log n$ . We can apply Lemma 2.3 since  $\beta < 2^{-d^2/4} = n^{-d/h} \leq |V_1|^{-d/h}$ . So there is a subset  $V_2 \subset V_1$  such that

$$|V_2| \geq \alpha^h m - 1 \geq \alpha^h m / 2 \geq (2t^4)^{-4d^{-1} \log n} \cdot \frac{n}{4k} \geq n^{1 - \frac{20 \log t}{d}} / (4k)$$

and every subset of  $V_2$  of size  $d$  has common neighborhood at least  $\beta n = 2^{-\binom{d}{2}} n$  in color  $i$ .

There are two possibilities: either every  $K_d$  in  $V_2$  is not rainbow, or there is a  $K_d$  in  $V_2$  that is rainbow. In the first case, the  $k$ -edge-coloring restricted to  $V_2$  is free of rainbow  $K_d$ , so

$$g(k, d) \geq |V_2| \geq n^{1 - \frac{20 \log t}{d}} / (4k).$$

Since  $n = g(k, t)$ , we can restate this inequality as

$$g(k, t) \leq 4k g(k, t)^{\frac{20 \log t}{d}} g(k, d).$$

In the second case, there is a rainbow  $d$ -tuple  $R \subset V_2$  such that  $N_i(R)$ , the common neighborhood of  $R$  in color  $i$ , has cardinality at least  $\beta n$ . The  $\binom{d}{2}$  colors present in  $R$  can not be present in  $N_i(R)$  since otherwise we would have a  $K_4$  using only two colors (the color  $i$  and the color that appears in both  $R$  and in  $N_i(R)$ ). In this case we have

$$g\left(k - \binom{d}{2}, t\right) \geq |N_i(R)| \geq \beta n = 2^{-\binom{d}{2}} g(k, t).$$

In either case we have

$$g(k, t) \leq \max\left(4k g(k, t)^{\frac{20 \log t}{d}} g(k, d), 2^{\binom{d}{2}} g\left(k - \binom{d}{2}, t\right)\right),$$

which completes the proof.  $\square$

Having finished all the necessary preparation, we are now ready to prove Theorem 1.1, which says that  $g(k) \leq 2^{2000k}$  for  $k > 2^{100}$ . The iterated logarithm  $\log^* n$  is defined by  $\log^* n = 0$  if  $n \leq 1$  and otherwise  $\log^* n = 1 + \log^* \log n$ . It is straightforward to verify that  $\log^* n < \log n$  holds for  $n > 8$ .

**Proof of Theorem 1.1:** Note that  $g(k) = g(k, k)$  since no  $k$ -edge-coloring contains a rainbow  $K_k$ . Assume  $k > 2^{100}$  and suppose for contradiction that there is a  $k$ -edge-coloring of  $K_n$  with  $n = g(k) \geq 2^{2000k}$  such that every  $K_4$  has at least three colors.

Let  $t_1 = k$ , and if  $t_i > 2^{100}$ , let  $t_{i+1} = (\log t_i)^2$ . We first exhibit several inequalities which we will use. We have  $t_{i+1} > 100 \log t_i$  and  $20 \frac{\log t_i}{t_{i+1}} = 20 / \log t_i \leq \frac{1}{5}$ . Let  $\ell$  be the largest positive integer for which  $t_\ell$  is defined, so  $100 < t_\ell \leq 2^{100}$ . Note that  $\ell < 2 \log^* k$  as one can easily check that  $t_{j+1} = (\log t_j)^2 = (2 \log \log t_{j-1})^2 < \log t_{j-1}$ . Since  $\ell < 2 \log^* k \leq 2 \log k$  and  $n \geq 2^{2000k}$ , then  $(4k)^\ell < n^{1/12}$ . We have  $\sum_{i=1}^{\ell-1} 20 / \log t_i < 1/4$  since the largest term in the sum is  $20 / \log t_{\ell-1} < 1/5$ , and  $20 / \log t_{\ell-i} < 2^{-5i}$  for  $2 \leq i \leq \ell - 1$ . Putting this together, we have

$$(4k)^{\ell-1} n^{\sum_{i=1}^{\ell-1} 20 / \log t_i} < n^{1/3}.$$

To get an upper bound on  $g(k, k)$  we repeatedly apply Lemma 2.5. Given  $k'$  and  $t = t_i$ , to bound  $g(k', t)$ , we use this lemma with  $d = t_{i+1}$ . Note that we have  $d = t_{i+1} > 100 \log t_i$ , so indeed the condition of the lemma holds. If  $k' < \binom{t_{i+1}}{2}$ , then  $g(k', t_i) = g(k', t_{i+1})$ . Otherwise, we have one of two possible upper bounds given by (1). If the maximum of the two terms in (1) is the left bound, then

$$g(k', t) \leq 4k' g(k', t)^{\frac{20 \log t}{d}} g(k', d) \leq 4kn^{\frac{20 \log t}{d}} g(k', d) = 4kn^{20 / \log t_i} g(k', d),$$

otherwise we have  $g(k', t) \leq 2^j g(k' - j, t)$  with  $j = \binom{d}{2}$ . Since  $\frac{g(k', t)}{g(k', d)} \leq 4kn^{20/\log t_i}$ , we can only accumulate up to a total upper bound factor of

$$\prod_{i=1}^{\ell-1} 4kn^{20/\log t_i} = (4k)^{\ell-1} n^{\sum_{i=1}^{\ell-1} 20/\log t_i} < n^{1/3}$$

in all of the applications of the left bound. When we use the right bound, we pick up a factor of  $\frac{g(k', t)}{g(k' - j, t)} \leq 2^j$  with  $j = \binom{d}{2}$  and also decrease  $k'$  by  $j$ . So this can only give another multiplicative factor of at most  $2^k$  in all of the applications of the right bound.

Notice that when we finish repeatedly applying Lemma 2.5 we end up with a term of the form  $g(k_0, t_\ell)$  with  $k_0 \leq k$ . In that case, we use that  $t_\ell \leq 2^{100}$  together with Lemma 2.4 to bound it by  $g(k, t_\ell) \leq t_\ell^{12k} \leq 2^{1200k}$ . Putting this all together, we obtain the upper bound

$$n = g(k) = g(k, k) < n^{1/3} 2^k g(k, t_\ell) < 2^{1201k} n^{1/3},$$

which implies that  $n < 2^{2000k}$ . This completes the proof.  $\square$

### 3 Monochromatic or Rainbow Cliques

In this section, we prove bounds on the smallest  $n$ , denoted by  $M(k, t, s)$ , such that every  $k$ -edge-coloring of  $K_n$  contains a monochromatic  $K_t$  or a rainbow  $K_s$ . The following proposition is a straightforward generalization of Lemma 2.4.

**Proposition 3.1** *We have  $M(k, t, s) \leq s^{4kt}$ .*

Let  $M_s(t_1, \dots, t_k)$  be the maximum  $n$  such that there is a  $k$ -edge-coloring of  $K_n$  with colors  $\{1, \dots, k\}$  without a rainbow  $K_s$  and without a monochromatic  $K_{t_i}$  in color  $i$  for  $1 \leq i \leq k$ . The above proposition follows from repeated application of the following recursive bound.

**Lemma 3.2** *We have*

$$M_s(t_1, \dots, t_k) \leq s^4 \max_{1 \leq i \leq k} M_s(t_1, \dots, t_i - 1, \dots, t_k).$$

**Proof.** By Lemma 2.1, for every edge-coloring of  $K_n$  without a rainbow  $K_s$ , there is a vertex  $v$  with degree at least  $n/s^4$  in some color  $i$ . If the coloring of  $K_n$  does not contain a monochromatic  $K_{t_i}$  in color  $i$ , then the neighborhood of  $v$  in color  $i$  has at least  $n/s^4$  vertices and does not contain  $K_{t_i-1}$  in color  $i$ , completing the proof.  $\square$

Using a slightly better estimate by Alon et al. [1] (which we mentioned earlier) instead of Lemma 2.1, one can improve the constant in the exponent of the above proposition from 4 to 3. Together with the next lemma, Proposition 3.1 determines  $M(k, t, 3)$  up to a constant factor in the exponent.

**Lemma 3.3** *For all positive integers  $k$  and  $t$  with  $k$  even and  $t \geq 3$ , we have  $M(k, t, 3) > 2^{kt/4}$ .*

**Proof.** To prove the lemma, it suffices by induction to prove  $M(k, t, 3) - 1 \geq 2^{t/2} (M(k - 2, t, 3) - 1)$  for all  $k \geq 2$  and  $t \geq 3$ . Consider a 2-edge-coloring  $C_1$  of  $K_m$  with  $m = 2^{t/2}$  and without a monochromatic  $K_t$ . Such a 2-edge-coloring exists by the well-known lower bound of Erdős [5] on the 2-color Ramsey number  $R(t; 2)$ . Consider also a  $(k - 2)$ -edge-coloring  $C_2$  of  $K_r$  with  $r = M(k - 2, t, 3) - 1$  without a rainbow triangle and without a monochromatic  $K_t$ . We use these two colorings to make a new edge-coloring  $C_3$  of  $K_{mr}$  with  $k$  colors: we first partition the vertices of  $K_{mr}$  into  $m$  vertex subsets  $V_1, \dots, V_m$  each of size  $r$ , and color any edge  $e = (v, w)$  with  $v \in V_i, w \in V_j$ , and  $i \neq j$  by the color of  $(i, j)$  in the 2-edge-coloring  $C_1$  of  $K_m$ , and color within each  $V_i$  identical to the coloring  $C_2$  of  $K_r$ . First we show that coloring  $C_3$  has no rainbow triangle. Indeed, consider three vertices of  $K_{mr}$ . If all three vertices lie in the same vertex subset  $V_i$ , then the triangle between them is not rainbow by the assumption on coloring  $C_2$ . If exactly two of the three vertices lie in the same vertex subset, then the two edges from these vertices to the third vertex will receive the same color. Finally, if they lie in three different vertex subsets, then the triangle between them receives only colors from  $C_1$  and is not rainbow since  $C_1$  is a 2-coloring. Similarly, one can see that coloring  $C_3$  has no monochromatic  $K_t$ , which completes the proof.  $\square$

## 4 Concluding Remarks

In this paper we proved that there exists a constant  $c$  such that every  $k$ -edge-coloring of  $K_n$  with  $n \geq 2^{ck}$  contains a  $K_4$  whose edges receive at most two colors. On the other hand, for  $n \leq 2^{c(\log k)^2}$ , Mubayi constructed a  $k$ -edge-coloring of  $K_n$  in which every  $K_4$  receives at least three colors. There is still a large gap between these results. We believe that the lower bound is closer to the truth and the correct growth is likely to be subexponential in  $k$ .

Our upper bound is equivalent to  $f(n, 4, 3) \geq (\log n)/4000$  for  $n$  sufficiently large. Kostochka and Mubayi showed that  $f(n, 2a, a + 1) \geq c_a \frac{\log n}{\log \log \log n}$ , where  $c_a$  is a positive constant for each integer  $a \geq 2$ . Like the Kostochka-Mubayi proof, our proof can be generalized to demonstrate that for every integer  $a \geq 2$  there is  $c_a > 0$  such that  $f(n, 2a, a + 1) \geq c_a \log n$  for every positive integer  $n$ . For brevity, we do not include the details.

We do not yet have a good understanding of how  $M(k, t, s)$ , which is the smallest positive integer  $n$  such that every  $k$ -edge-coloring of  $K_n$  has a monochromatic  $K_t$  or a rainbow  $K_s$ , depends on  $s$ . From the definition, it is an increasing function in  $s$ . For constant  $s$ , we showed that  $M(k, t, s)$  grows only exponentially in  $k$ . On the other hand, for  $\binom{s}{2} > k$ , we have  $M(k, t, s) = R(t; k)$ , so understanding the behavior of  $M(k, t, s)$  for large  $s$  is equivalent to understanding the classical Ramsey numbers  $R(t; k)$ .

## References

- [1] N. Alon, T. Jiang, Z. Miller and D. Pritikin, Properly colored subgraphs and rainbow subgraphs in edge-colorings with local constraints, *Random Structures and Algorithms* **23** (2003), 409-433.
- [2] M. Axenovich, A Generalized Ramsey problem, *Discrete Math.* **222** (2000), 247-249.



- [3] M. Axenovich, Z. Füredi, and D. Mubayi, On generalized Ramsey theory: the bipartite case, *J. Combin. Theory Ser. B* **79** (2000), 66–86.
- [4] M. Axenovich and P. Iverson, Edge-colorings avoiding rainbow and monochromatic subgraphs, to appear in *Discrete Math.*
- [5] P. Erdős, Some remarks on the theory of graphs, *Bull. Amer. Math. Soc.* **53** (1947), 292–294.
- [6] P. Erdős, Solved and unsolved problems in combinatorics and combinatorial number theory, *Congressus Numerantium* **32** (1981), 49–62.
- [7] P. Erdős and A. Gyárfás, A variant of the classical Ramsey problem, *Combinatorica* **17** (1997), 459–467.
- [8] J. Fox and B. Sudakov, Density theorems for bipartite graphs and related Ramsey-type results, submitted.
- [9] R. Graham, B. Rothschild, and J. Spencer, *Ramsey Theory*, 2<sup>nd</sup> ed., Wiley, New York, 1990.
- [10] A. Kostochka and D. Mubayi, When is an almost monochromatic  $K_4$  guaranteed?, submitted.
- [11] A. Kostochka and V. Rödl, On graphs with small Ramsey numbers. *J. Graph Theory* **37** (2001), 198–204.
- [12] D. Mubayi, Edge-coloring cliques with three colors on all 4-cliques, *Combinatorica* **18** (1998), 293–296.
- [13] D. Mubay, An explicit construction for a Ramsey problem, *Combinatorica* **24** (2004), 313–324.
- [14] G. N. Sárközy, and S. M. Selkow, On edge colorings with at least  $q$  colors in every subset of  $p$  vertices, *Electron. J. Combin.* **8** (2001), Research Paper 9, 6 pp.
- [15] G. N. Sárközy, and S. M. Selkow, An application of the regularity lemma in generalized Ramsey theory, *J. Graph Theory* **44** (2003), 39–49.
- [16] B. Sudakov, Few remarks on the Ramsey-Turan-type problems, *J. Combinatorial Theory Ser. B* **88** (2003), 99–106.