Turán-type results for partial orders and intersection graphs of convex sets

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Dedicated to András Hajnal on the occasion of his 75th birthday.

Abstract

We prove Ramsey-type results for intersection graphs of geometric objects the plane. In particular, we prove the following bounds, all of which are tight apart from the constant c. There is a constant $c > 0$ such that for every family $\mathcal{F}$ of n convex sets in the plane, the intersection graph of $\mathcal{F}$ or its complement contains a balanced complete bipartite graph of size at least cn. There is a constant $c > 0$ such that for every family $\mathcal{F}$ of n x-monotone curves in the plane, the intersection graph G of $\mathcal{F}$ contains a balanced complete bipartite graph of size at least $cn/\log n$ or the complement of G contains a balanced complete bipartite graph of size at least cn. Our bounds rely on new Turán-type results on incomparability graphs of partially ordered sets.

1 Introduction

A classic result of Erdős and Szekeres [10] in Ramsey theory states that every graph on n vertices contains a clique or an independent set of size[1] at least $\frac{1}{2}\log n$. This bound is tight up to a constant factor: Erdős [7] showed that there exists a graph on n vertices, for every integer $n > 1$, with no clique or independent set of more than $2\log n$ vertices. Erdős and Hajnal [8] proved that certain graphs contain much larger cliques or independent sets: For every hereditary family $\mathcal{F}$ of graphs other than the family of all graphs, there is a constant $c(\mathcal{F}) > 0$ such that every graph in $\mathcal{F}$ with n vertices contains a clique or an independent set of size at least $e^{c(\mathcal{F})\sqrt{\log n}}$. (A family of graphs is *hereditary* if it is closed under taking induced subgraphs.) They also asked whether this bound can be improved to $n^{c(\mathcal{F})}$.

A complete bipartite graph, whose vertex classes are of the same size or their sizes differ by at most *one*, is said to be balanced. A balanced complete bipartite graph with n vertices is called a *bi-clique of size n*. The problem of Erdős and Hajnal motivates the definition of the following two properties of a family $\mathcal{F}$ of graphs: We say that

1. $\mathcal{F}$ has the *(weak) Erdős-Hajnal property* if there is a constant $c(\mathcal{F}) > 0$ such that every graph in $\mathcal{F}$ on n vertices contains a clique or an independent set of size $n^{c(\mathcal{F})}$.

2. $\mathcal{F}$ has the *strong Erdős-Hajnal property* if there is a constant $b(\mathcal{F}) > 0$ such that every graph $G \in \mathcal{F}$ with $n > 1$ vertices or its complement $\overline{G}$ contains a bi-clique of size $b(\mathcal{F})n$.

Alon *et al.* [1] proved that if a hereditary family of graphs has the strong Erdős-Hajnal property, then it also has the Erdős-Hajnal property. For partial results on the Erdős-Hajnal problem, see [2], [3], [4], and [9].

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[1] All logarithms in this paper are of base two.

The *intersection graph* of a set system is a graph whose vertices are in one-to-one correspondence with the sets, with two vertices being connected by an edge if and only if the corresponding sets have at least one element in common. As noted by Ehrlich, Even, and Tarjan [6], not every graph can be realized as the intersection graph of connected sets in the plane. For instance, the bipartite graph on 15 vertices formed by replacing each edge of K_5 by a path of length 2 has no such realization. This implies, using the above result of Erdős and Hajnal, that the intersection graph of any n connected sets in the plane contains a clique or an independent set of size $e^{c\sqrt{\log n}}$, for some absolute constant $c > 0$. This general bound has been improved for families of intersection graphs of certain geometric objects in the plane.

Pach and Solymosi [16] proved that the family of intersection graphs of line segments in the plane has the strong Erdős-Hajnal property. Later, Alon *et al.* [1] generalized this result to intersection graphs of d-dimensional semialgebraic sets of description complexity at most D, for any fixed positive integers d and D.

In this paper, we prove similar results for intersection graphs of *convex sets* and *x-monotone curves* (that is, continuous curves in the plane such that every line parallel to the y-axis intersects each of them in at most one point). A common feature of these objects is that the boundaries of two convex sets, as well as two x-monotone curves, may intersect in an arbitrary number of points, in sharp contrast to semialgebraic sets in "general position."

Theorem 1 *The family of intersection graphs of convex sets in the plane has the strong Erdős-Hajnal property. That is, there exists a constant $c > 0$ with the property that the intersection graph G of any collection of n convex sets contains a bi-clique of size cn, or its complement $\overline{G}$ contains a bi-clique of size cn.*

The (weak) Erdős-Hajnal property for the family of intersection graphs of compact convex sets in the plane has been established by Larman *et al.* [15, 18]. For the bipartite version, the best previous result [12] was that the intersection graph G of any collection of n compact convex sets in the plane, or its complement $\overline{G}$, contains a bi-clique of size $n^{1-o(1)}$.

Theorem 1 does not generalize to higher dimensions: Tietze [21] showed that *every* graph can be realized as the intersection graph of convex compact sets in $\mathbb{R}^3$.

Theorem 2 *There exists a constant $c > 0$ with the property that the intersection graph G of any collection of n x-monotone curves in the plane satisfies at least one of the following two conditions:*

(a) G contains a bi-clique of size $\frac{cn}{\log n}$; or

(b) $\overline{G}$, the complement of G, contains a bi-clique of size cn.

The last theorem easily generalizes to *vertically convex* objects, that is, to connected sets with the property that every vertical line intersects each of them in a connected interval, which may consist of just one point or may be empty. To see this, notice that for every finite collection of vertically convex objects in the plane, one can construct a collection of x-monotone curves with the same intersection graph: Pick a "witness" point in the intersection of each intersecting pair of objects, and within each object connect all witness points by a vertically convex curve. Slightly perturbing the picture, if necessary, we can ensure that none of these curves contains a whole vertical segment, that is, the curves are x-monotone.

The *comparability graph (incomparability graph)* of a partially ordered set, in short, *poset*, $(P, \prec)$ is a graph defined on the vertex set P so that two elements of P are adjacent if and only if they are comparable (incomparable). Every partially ordered set is the intersection of its linear extensions. The *dimension* of a poset is the minimum number of its linear extensions whose intersection is that poset.

One may wonder whether condition (a) in Theorem 2 can be replaced by the stronger property that G contains a bi-clique of size *cn*. This is not the case: It is easy to check [17, 19] that every incomparability graph is isomorphic to the intersection graph of x-monotone curves (in fact, continuous real functions defined on $[0, 1]$). Using this observation, a construction of Fox [11] shows that Theorem 2 is the best possible.

The proofs of Theorems 1 and 2 crucially depend on Turán-type results for incomparability graphs. Turán's classic problem is to determine $ex(n, H)$, the maximum number of edges that a graph with n vertices can have without containing a (not necessarily induced) subgraph isomorphic to H.

Let $\mathcal{C}$ and $\mathcal{I}$ denote the families of comparability graphs and incomparability graphs. For any d, let $\mathcal{C}_d$ and $\mathcal{I}_d$ denote the families of comparability graphs and incomparability graphs of dimension d. Furthermore, let

$$\mathrm{ex}_{\mathcal{C}}(n,H) = \max\{|E(G)| : G \in \mathcal{C},\ H \not\subseteq G, \text{ and } |V(G)| = n\},$$

and define the functions $\mathrm{ex}_{\mathcal{C}_d}(n,H)$, $\mathrm{ex}_{\mathcal{I}}(n,H)$, and $\mathrm{ex}_{\mathcal{I}_d}(n,H)$ analogously.

If the excluded graph H is a clique, according to Turán's theorem [22], $\mathrm{ex}(n,K_t)$ is attained for the balanced complete $(t-1)$-partite graph with n vertices. Since every $(t-1)$-partite complete graph is both a comparability graph and an incomparability graph, we obtain that $\mathrm{ex}(n,K_t) = \mathrm{ex}_{\mathcal{C}}(n,K_t) = \mathrm{ex}_{\mathcal{I}}(n,K_t)$, for all $n,t \geq 2$.

On the other hand, if the excluded graph is a bi-clique, Turán's questions, when restricted to comparability and incomparability graphs, have very different answers than the "unrestricted" versions.

In Section 2, we establish the following two results, needed for the proofs of Theorems 1 and 2.

Theorem 3 *The maximum number of edges of a $K_{t,t}$-free (in)comparability graph of a 2-dimensional poset with n elements satisfies*

$$ex_{\mathcal{I}_2}(n,K_{t,t}) = ex_{\mathcal{C}_2}(n,K_{t,t}) \leq 2(t-1)n - \binom{2t-1}{2},$$

for every $t \geq 2$ and $n \geq 2t-1$.

Theorem 4 *There is a constant $c > 0$ such that for every $\delta > 0$ and $n \in \mathbb{N}$, we have*

$$ex_{\mathcal{I}}(n,K_{t,t}) < \delta n^2, \quad \text{where } t = \left\lfloor \frac{c\delta n}{\log\frac{1}{\delta}\log n} \right\rfloor.$$

In other words, if a poset P on n vertices has at least δn^2 incomparable pairs, then its incomparability graph contains a bi-clique of size $\Omega(\delta n/(\log\frac{1}{\delta}\log n))$. Note that the size of the largest bi-clique in a random graph with n vertices and δn^2 edges (and in its complement) is almost surely $O_\delta(\log n)$, for any $0 < \delta < 1/2$.

In Section 3, we establish an analogue of Theorem 4 for comparability graphs of posets (Theorem 7). It is not needed for the proof of Theorems 1 and 2, but it enables us to strengthen a theorem of Fox [11] (see Theorem 8).

It is very easy to see that it is sufficient to establish Theorems 1 and 2 for collections of sets intersecting the same line. To deal with such collections, in Sections 4 and 5 we develop some auxiliary results (Lemmas 10, 13, and 14) for "flags" and "bridges," that is, for connected sets that are incident to one line or lie between two parallel lines, respectively. One is designed to address the case when the average degree of the vertices in the intersection graph G is smaller than $\varepsilon|V(G)|$, for a suitable constant $\varepsilon \in (0,1)$, while the other two analyze the opposite situation. In the first case, we show the existence of a large bi-clique in the complement of G, and in the latter ones, in G itself. In these latter cases, we use the Turán-type results for incomparability graphs, established in Section 2.

The pieces of the proofs of Theorems 1 and 2, following the above strategy, are put together in Section 6. The last section contains a few remarks and open problems.

2 Turán-type results for incomparability graphs

The aim of this section is to prove Theorems 3 and 4.

Recall that $ex_{\mathcal{C}_d}(n,K_{t,t})$ (and $ex_{\mathcal{I}_d}(n,K_{t,t})$) is the maximum number of edges that a $K_{t,t}$-free graph of n vertices can have if it is the comparability (incomparability, resp.) graph of a d-dimensional partial order. We call a graph G *r-degenerate* if every subgraph of G contains a vertex of degree at most r. Clearly, the number of edges of any r-degenerate graph G with $n > r$ vertices satisfies

$$|E(G)| \leq rn - \binom{r+1}{2},$$

and this bound is tight.

We use the notation $[n] = \{1, \ldots, n\}$. For any permutation π of $[n]$, let $P_\pi = ([n], <_\pi)$ denote the 2-dimensional partial order on $[n]$, in which $i <_\pi j$ if and only if $i < j$ and $\pi(i) < \pi(j)$.

Proof of Theorem 3: It is sufficient to prove the statement for incomparability graphs, because the corresponding statement for comparability graphs follows by the simple observation that $\mathcal{C}_2 = \mathcal{I}_2$. Further, it is enough to show that the incomparability graph of every 2-dimensional partial order is $(2t-2)$-degenerate.

Every 2-dimensional poset of n vertices can be realized as P_π, for a suitable permutation π. Suppose for contradiction that the degree of every vertex of the incomparability graph P_π is at least $2t-1$. Notice that every $i \in [n]$ is incomparable with at most $i - 1 + \pi(i) - 1$ other elements of $[n]$. Since each element $i \in [n]$ is incomparable with at least $2t-1$ other elements of P_π, we have $\pi(i) \geq t+1$ for $i \in [t]$ and $i \geq t+1$ for $\pi(i) \in [t]$. In particular, every $i \in [t]$ is incomparable with every element j with $\pi(j) \in [t]$. Hence, the incomparability graph contains $K_{t,t}$, which is a contradiction. $\Box$

The bound in Theorem 3 is roughly within a factor of two from the truth. To see this, consider the following simple construction. Let $n = \ell(2t-1)$, for some $\ell \in \mathbb{N}$, and let P_π denote the 2-dimensional poset defined by the permutation $\pi(i + k(2t-1)) = 2t - i + k(2t-1)$, for $1 \leq i \leq 2t-1$ and $0 \leq k \leq \ell - 1$. The incomparability graph of P_π is the disjoint union of cliques of size $2t-1$, hence it is $K_{t,t}$-free and $(2t-2)$-regular, so that its number of edges is $(t-1)n$.

Corollary 5 *For all positive integers d, n, and t with $n \geq 2t-1$, we have*

$$ex_{\mathcal{I}_d}(n, K_{t,t}) \leq (d-1)\left(2(t-1)n - \binom{2t-1}{2}\right).$$

Proof: Consider a d-dimensional poset $(P, \prec)$ whose incomparability graph does not contain $K_{t,t}$ as a subgraph. We may assume that $P = [n]$ and that there are permutations $\pi_1, \ldots, \pi_d$ of P with π_1 being the identity permutation such that $i \prec j$ if and only if $\pi_k(i) < \pi_k(j)$ for every $k \in [d]$.

Two elements, i and j with $i < j$, are incomparable if and only if there is an index $k \in [2, d]$ such that $\pi_k(i) > \pi_k(j)$. Hence, the number of edges of the incomparability graph of $(P, \prec)$ is at most the sum of the number of edges in the $d-1$ incomparability graphs of the 2-dimensional partially ordered sets $P_{\pi_2}, \ldots, P_{\pi_d}$. For $k \in [2, d]$, the incomparability graph of P_{π_k} does not contain $K_{t,t}$, since otherwise the incomparability graph of $(P, \prec)$ contains $K_{t,t}$. By Theorem 3, the incomparability graph of $(P, \prec)$ has at most $(d-1)\left(2(t-1)n - \binom{2t-1}{2}\right)$ edges. $\Box$

It is a simple corollary to Dilworth' theorem [5] that every partially ordered set on n elements contains a chain or an antichain of size at least $\sqrt{n}$. For the proof of Theorem 4, we need the following bipartite analogue of this result.

Lemma 6 (Fox [11]) *If n is sufficiently large, then every poset of n elements contains two disjoint subsets A and B, each of size at least $\frac{n}{4\log n}$, such that either every element of A is larger than every element of B or every element of A is incomparable with every element of B.*

Given a poset $(P, <)$, for any $x \in P$ and for any subset $S \subseteq P$, define $D_S(x)$, the *down-set* of x in S, as the set of elements in S below x. That is, let $D_S(x) = \{s : s \in S \text{ and } s < x\}$. Analogously, let the *up-set* of x in S be defined as $U_S(x) = \{s : s \in S \text{ and } x < s\}$.

Proof of Theorem 4: Let $(P, <)$ be a poset with n elements, whose incomparability graph contains no $K_{t,t}$. Let $<^*$ be a linear extension of $<$, let X and Y denote the set of the top $\lfloor \frac{n}{2} \rfloor$ and the set of the bottom $\lceil \frac{n}{2} \rceil$ elements of P with respect to $<^*$. Clearly, we have $P = X \cup Y$. Let X_1 be the set of all $x \in X$ with $|D_X(x)| \geq t$, and let $X_2 = X \setminus X_1$ be its complement. Similarly, let Y_1 be the subset of all $y \in Y$ with $|U_Y(y)| \geq t$, and let $Y_2 = Y \setminus Y_1$.

Every $x \in X_1$ is comparable with every $y \in Y_1$, otherwise every element of $D_X(x)$ is incomparable with every element of $U_Y(y)$, which means that the incomparability graph of $(D_X(x) \cup D_Y(y), <)$ already contains a $K_{t,t}$.

For every $x \in X_2$, the down-set of x in X is smaller than t, and so the *comparability* graph of $(X_2, <)$ contains no $K_{t,t}$. By Lemma 6, however, the comparability graph or the incomparability graph of $(X_2, <)$ contains $K_{s,s}$ with $s = \frac{|X_2|}{4 \log |X_2|}$, provided that $|X_2|$ is sufficiently large. Hence, $t > \frac{|X_2|}{4 \log |X_2|}$ if $|X_2|$ is sufficiently large; and likewise, $t > \frac{|Y_2|}{4 \log |Y_2|}$ if $|Y_2|$ is sufficiently large. Since $\log |X_2| \leq \log n$ and $\log |Y_2| < \log n$, it follows that for every sufficiently large n, we have $|X_2| \leq 4t \log n$ and $|Y_2| \leq 4t \log n$. Every element of P is incomparable with at most $n-1$ other elements of P, and so, for n sufficiently large, the elements of X_2 and Y_2 participate in at most $8t(n-1)\log n$ incomparable pairs of P. Since there is no incomparable pair (x, y) with $x \in X_1$ and $y \in Y_1$, we have

$$ex_{\mathcal{I}}\left(n, K_{t,t}\right) \leq 8t(n-1)\log n + 2ex_{\mathcal{I}}\left(\left\lceil \frac{n}{2} \right\rceil, K_{t,t}\right),$$

if n is large enough. Using the fact that every graph with m vertices has at most $\binom{m}{2}$ edges, after iterating the above inequality j times, we obtain

$$ex_{\mathcal{I}}\left(n, K_{t,t}\right) = O\left(jtn \log n\right) + \frac{n^2}{2^j}.$$

Setting $j := \lceil \log \frac{1}{\delta} \rceil + 1$, the theorem follows. $\square$

3 Turán-type results for comparability graphs

In this section, we estimate the function $ex_{\mathcal{C}}(n, K_{t,t})$, the maximum number of edges that a $K_{t,t}$-free comparability graph with n vertices can have.

Instead of studying $ex_{\mathcal{C}}(n, K_{t,t})$ directly, it will be more convenient to bound its inverse. Let $T(n, m)$ denote the largest integer t such that every comparability graph with n vertices and at least m edges contains $K_{t,t}$. The following theorem demonstrates a dramatic change in $T(n, m)$, when m is roughly $n^2/4$.

Theorem 7

(1) For every $\varepsilon > 0$, there is a constant $c(\varepsilon)$ such that $T(n, (\frac{1}{4} - \varepsilon)n^2) \leq c(\varepsilon) \log n$.

(2) There are constants $c_1 > 0$ and $c_2 > 0$ such that $c_1\sqrt{n} \leq T(n, \frac{n^2}{4}) \leq c_2\sqrt{n \log n}$.

(3) For every $\varepsilon > 0$, we have $T(n, (\frac{1}{4} + \varepsilon)n^2) \geq \frac{\varepsilon}{2} n$.

Proof: We first prove the upper bounds on $T(n, m)$. Note that every bipartite graph is a comparability graph. Consider a random bipartite graph with $\lceil \frac{n}{2} \rceil$ vertices in the first class and $\lfloor \frac{n}{2} \rfloor$ vertices in the second class, there is an edge between any two vertices independently at random with probability p. Letting $p = 1 - \varepsilon$, it is an easy exercise to show that with positive probability this random bipartite graph has at least $(\frac{1}{4} - \varepsilon)n^2$ edges and contains no bi-clique of size $c(\varepsilon) \log n$ for some constant $c(\varepsilon)$. This proves (1).

Let P be the poset on $n > 1$ elements which has $\lceil \sqrt{n \log n} \rceil$ elements that form a chain and are larger than the $n - \lceil \sqrt{n \log n} \rceil$ remaining elements, and the comparability graph of the remaining $n - \lceil \sqrt{n \log n} \rceil$ elements is a random bipartite graph with at least $\lfloor \frac{n - \lceil \sqrt{n \log n} \rceil}{2} \rfloor$ elements in each of its classes and each edge taken randomly with probability $p = 1 - \sqrt{\frac{\log n}{n}}$. It is easy to check that with positive probability, the comparability graph G of the poset P has at least $n^2/4$ edges and the largest bi-clique in G has $O(\sqrt{n \log n})$ vertices. This establishes the upper bound in (2).

We next prove the lower bounds on $T(n, m)$. Let P be a partially ordered set with n elements such that its comparability graph G does not contain $K_{t,t}$. Let X be the subset of P consisting of the elements $x \in X$ with $|D_P(x)| \geq t$, and let Y be the subset of P consisting of the elements $y \in Y$ with $|U_P(y)| \geq t$. Let Z denote $P \setminus (X \cup Y)$. The sets X and Y are disjoint, for if $x \in X \cap Y$, then $U_P(x)$ and $D_P(x)$ each have at least t elements and every element of $U_P(x)$ is larger than every element of $D_P(x)$, a contradiction. Hence, $P = X \cup Y \cup Z$ is a partition of P. Every element of Z belongs to at most $2t - 2$ comparable pairs. Every element $x \in X$ has fewer than t elements above x in X and every element $y \in Y$ has fewer than t elements below y in Y. Hence, the number of edges of G is at most

$$(2t-2)|Z| + (t-1)(|X|+|Y|) + |\{(x,y) : x \in X, y \in Y \text{ and } x \text{ and } y \text{ are comparable }\}|.$$

Therefore, the number of edges of G satisfies

$$|E(G)| < 2tn + |\{(x,y) : x \in X, y \in Y \text{ and } x \text{ and } y \text{ are comparable }\}|.$$

If the number of edges of G is at least $\frac{n^2}{4}$, then the number of comparable pairs between X and Y is more than $\frac{n^2}{4} - 2nt$. If $t = o(n)$, then this yields that X and Y are roughly of size $\frac{n}{2}$. The number of elements in X that are connected to all but at most $5t$ elements in Y is larger than $n/4$, provided that n is sufficiently large. Pick t such elements and delete all the (at most $5t^2$) vertices in Y that are *not* connected to at least one of them. As long as $5t^2 < n/4$, we are still left with at least t points, showing that $G \supset K_{t,t}$ with $t = \Omega(\sqrt{n})$. This proves the lower bound in (2).

The number of comparable pairs between X and Y is at most $n^2/4$. If the number of edges of G is at least $(\frac{1}{4}+\varepsilon)n^2$, then $2tn \geq \varepsilon n^2$, hence $t \geq \frac{\varepsilon n}{2}$. This proves (3). $\square$

Combining Theorem 4 and part (3) of Theorem 7, we obtain the following result, which is a strengthening of Lemma 6.

Theorem 8 *There is a positive constant c such that every comparability graph G on $n > 1$ vertices contains a bi-clique of size at least cn, or its complement $\overline{G}$ contains a bi-clique of size at least $\frac{cn}{\log n}$.*

To see that Theorem 8 is indeed a strengthening of Lemma 6, we need the simple observation that if the comparability graph of a poset P contains $K_{t,t}$, then there are two subsets A and B of P with $|A| = |B| \geq t/2$ such that every element of A is greater than every element of B [11].

In view of the fact that every comparability graph is the complement of an intersection graph of x-monotone curves [17, 19], Theorem 8 is also a direct corollary of Theorem 2. This is not too surprising, because the proof of Theorem 2 heavily relies on Theorem 4, which is the main component of the proof of Theorem 8.

4 Connected sets met by a line

The aim of this section is to establish the existence of a bi-clique of linear size in the complement of the intersection graph G of a system of n connected sets that intersect a line L, provided that G is relatively sparse. For technical reasons, it will be simpler to prove such a result first for a *collection of flags*, which is defined as a collection of sets, each lying in the closed halfplane bounded by a line L and having at least one common point with L. Once we have proved the result for this case, the general statement is an easy corollary (see Theorem 11 below).

Consider a set S of n flags on the right of a vertical line L. Introduce a linear order on S along L, as follows: For each $\alpha \in S$, fix a point $p(\alpha) \in \alpha \cap L$ and choose a linear order $\prec$ such that $\alpha \prec \beta$ if the y-coordinate of $p(\alpha)$ is less than or equal to that of $p(\beta)$. Label the elements of S with the numbers $1, \ldots, n$ according to this order. Define the *distance* between two flags in S as the cyclic distance between their labels. That is, the distance between the flags of label i and label j is $\min(|i-j|, n-|i-j|)$. For $a, b \in \mathbb{Z}_n$, we define the *cyclic interval* $[a,b] := \{a, a+1, \ldots, b\}$. Note that, if $n > 2$, then according to this definition, $[a,b] \neq [b,a]$ for $a \neq b$.

Lemma 9 *If every element of a system S of $n > 1$ flags intersects at most $n/12$ others, then there are disjoint subsets $A, B \subset S$ with $|A| = |B| \geq \frac{n}{6}$ such that every flag in A is disjoint from every flag in B.*

Proof: Assume without loss of generality that the line L "holding" the flags is vertical. We distinguish two cases.

Case 1: There are two intersecting flags, α and β, at distance at least $\lfloor n/3 \rfloor$.

Denote the labels of α and β by a and b. Let $A \subset S$ (and $B \subset S$) be the set of flags which do not intersect either α or β and whose labels lie in the cyclic interval $[a, b]$ ($[b, a]$, respectively). Fig. 1, left, depicts an example where each flag is a simple curve.

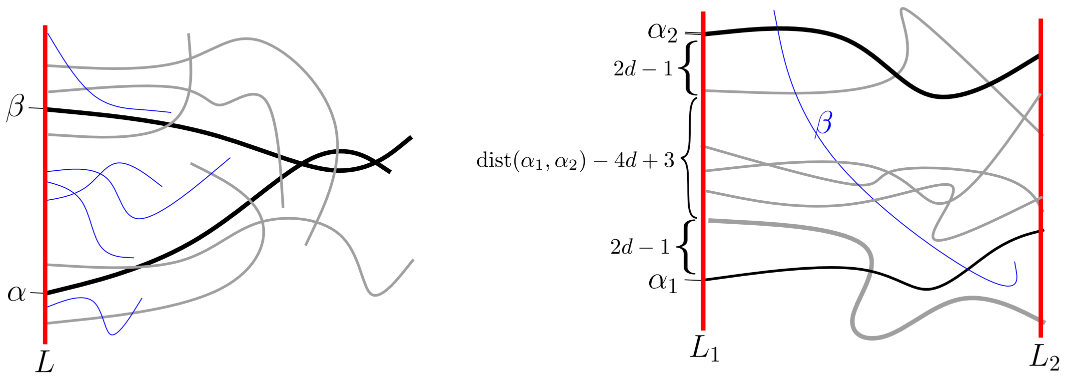

Figure 1: On the left: Flags for a line L. Flags α and β at distance at least $\lfloor n/3 \rfloor$ are bold, flags intersecting α or β are grey. On the right: Bridges between lines L_1 and L_2, and a connected set β. If β intersects both α_1 and α_2, then it must intersect all bridges that lie between α_1 and α_2.

Both A and B contain at least $(\lfloor \frac{n}{3} \rfloor - 1) - 2(\frac{n}{12} - 1)$ flags, since α and β each intersect at most $\frac{n}{12}$ other flags. Hence, $|A|, |B| \geq \frac{n}{6}$ and no flag in A intersects any flag in B.

Case 2: No two flags in S at distance at least $\lfloor n/3 \rfloor$ intersect.

Let A and B be the sets of flags whose labels belong to the intervals $[1, \lceil \frac{n}{6} \rceil]$ and $[\lceil \frac{n}{2} \rceil, \lceil \frac{2n}{3} \rceil]$, respectively. Every flag in A is disjoint from every flag in B, and the cardinality of each of A and B is at least $\frac{n}{6}$. $\square$

Lemma 10 *Let G be the intersection graph of a collection of n flags. If the average degree of a vertex in G is at most $n/24$, then the complement of G contains a bi-clique of size $2\lceil n/12 \rceil$.*

Proof: Delete successively the maximal degree vertices of the intersection graph G, until the degree of every remaining vertex is at most $n/24$. No more than $n/2$ vertices have been deleted, since $|E(G)| \leq n^2/48$. There are at least $n/2$ vertices left, each of degree at most $n/24$. Thus, we can apply Lemma 9 to the remaining intersection graph, to obtain that its complement, and hence $\overline{G}$, contains a bi-clique with $\lceil n/12 \rceil$ vertices in each of its vertex classes. $\square$

By successive application of the last statement, we obtain its analogue for any collection of sets met by a line L, which are not necessarily contained in one of the half-planes bounded by L.

Theorem 11 *Let G be the intersection graph of a collection S of $n > 1$ connected sets, each of which intersects a vertical line L in a nonempty interval. If the average degree of the vertices in G is less than $n/12^8$, then the complement of G contains a bi-clique of size at least $2\lceil 2n/12^5 \rceil$.*

We need the following easy technical lemma by Fox and Pach.

Lemma 12 (Fox and Pach [12]) *Let $S = S_1 \cup \ldots \cup S_m$ be a partition of a set S with $|S_i| = \ell$ for $1 \leq i \leq m$, and let A and B be disjoint subsets of S of the same size. For $1 \leq i \leq m$, let $A_i = A \cap S_i$ and $B_i = B \cap S_i$. Then there is a partition of the set $\{1, \ldots, m\}$ into two parts, I_1 and I_2, such that*

$$\sum_{i \in I_1} |A_i| \geq \frac{|A| - \ell}{2} \quad \textit{and} \quad \sum_{i \in I_2} |B_i| \geq \frac{|A| - \ell}{2}.$$

Proof of Theorem 11: Let L^- and L^+ denote the closed half-planes to the left and to the right of L. Note that the portions of the members of S lying in L^- (in L^+) form a system of flags. The average degree of the vertices in G is small enough so that we can use Lemma 10 successively. Applying Lemma 10 four times to the portions of the sets clipped in L^-, we obtain disjoint subsets $S_1, \ldots, S_{16} \subset S$ such that $|S_i| = \lceil n/12^4 \rceil$ and for any two sets $\alpha \in S_i$ and $\beta \in S_j$, $i \neq j$, the portions $\alpha \cap L^-$ and $\beta \cap L^-$ are disjoint. Applying Lemma 10 to the portions of the sets in $S' = \bigcup_{i=1}^{16} S_i$ clipped in L^+, there are disjoint subsets $A, B \subset S'$ such that $|A| = |B| = \lceil |S'|/12 \rceil$, and for any two sets $\alpha \in A$ and $\beta \in B$, the portions $\alpha \cap L^+$ and $\beta \cap L^+$ are disjoint. By Lemma 12, there are subsets $A' \subset A$ and $B' \subset B$ such that $|B'| = |A'| \geq (|A| - |S_1|)/2 \geq 2n/12^5$ and every element of A' is disjoint from every element of B', completing the proof. $\square$

5 Connected sets between two parallel lines

The results in this section will be used in the proof of both Theorems 1 and 2 to find a linear size bi-clique (or an almost linear size bi-clique) in the intersection graph G of a family of connected sets lying between two vertical lines, provided that G is relatively dense.

Two vertical lines, $L_1 : x = a$ and $L_2 : x = b$, determine a *vertical strip*, which is the closed region $R = \{p \in \mathbb{R}^2 : a \leq x(p) \leq b\}$ between the two lines. A *bridge* between two lines is a connected set that intersects both. For a bridge α between L_1 and L_2, let $y_1(\alpha) \in \mathbb{R} \cup \{-\infty\}$ be the infimum of the y-coordinates of all points of $\alpha \cap L_1$. Similarly, let $y_2(\alpha) \in \mathbb{R} \cup \{-\infty\}$ be the infimum of the y-coordinates of all points of $\alpha \cap L_2$. For a finite set A of bridges between L_1 and L_2, we can choose two linear orders on A such that $\alpha \prec_1 \beta$ if $y_1(\alpha) \leq y_1(\beta)$; and $\alpha \prec_2 \beta$ if $y_2(\alpha) \leq y_2(\beta)$. The intersection of $\prec_1$ and $\prec_2$ is a 2-dimensional partial order, which we denote by $\prec$.

The following lemma focuses on the intersections of bridges and other connected sets in a vertical strip between two parallel lines.

Lemma 13 *Let $0 < \varepsilon < \frac{1}{3}$, let A be a collection of at most n bridges between two vertical lines L_1 and L_2, and let B be a collection of at most εn connected sets that lie in the closed strip between L_1 and L_2, and that satisfy the conditions*
(1) the intersection graph of $A \cup B$ has at least $25\varepsilon^2 n^2$ edges,
(2) the intersection graph of A has at most $4\varepsilon^2 n^2$ edges.

Then there exist subsets $A'' \subset A$ and $B'' \subset B$ of size $|A''| = |B''| \geq \varepsilon^2 n$ such that every element of A'' intersects every element of B''.

Proof: Since B has at most εn elements, the number of intersecting pairs in B is at most $\binom{\varepsilon n}{2} < \varepsilon^2 n^2$. Thus, by conditions (1) and (2), there are at least $20\varepsilon^2 n^2$ intersecting pairs in $A \times B$. Let $d := \varepsilon n$, and let A' be the set of bridges in A intersecting fewer than d other elements of A. The size of $A \setminus A'$ is at most $8\varepsilon^2 n^2/d \leq 8\varepsilon n$, so the bridges in $A \setminus A'$ altogether may be involved in at most $|A \setminus A'| \cdot |B| \leq 8\varepsilon^2 n^2$ intersecting pairs that belong to $A \times B$. Hence, all the remaining at least $20\varepsilon^2 n^2 - 8\varepsilon^2 n^2 = 12\varepsilon^2 n^2$ intersecting pairs in $A \times B$ belong to $A' \times B$.

Let B' denote the set of all elements of B that intersect at least $5d = 5\varepsilon n$ bridges in A'. The number of intersecting pairs in $A' \times B'$ is at least

$$12\varepsilon^2 n^2 - 5d|B| \geq 7\varepsilon^2 n^2. \tag{1}$$

Next we show that there are subsets $A'' \subset A'$ and $B'' \subset B'$, each of size at least $\varepsilon^2 n$, such that every element in A'' intersects every element of B''.

Label the elements of A' with $1, 2, \ldots, |A'|$ according to the linear order $\prec_1$, and define the *distance* between two bridges in A' as the difference between their labels. If two bridges $\alpha_1, \alpha_2 \in A'$ with $\alpha_1 \prec_1 \alpha_2$ intersect, then every $\alpha \in A'$ such that $\alpha_1 \prec_1 \alpha \prec_1 \alpha_2$ intersects α_1 or α_2. So if the distance of an intersecting pair in A' is at least $2d$, then α_1 or α_2 intersects at least d bridges in A', contradicting the choice of set A'. Therefore, the distance between any two intersecting bridges in A' is at most $2d - 1$.

If $\beta \in B'$ intersects $\deg\beta \geq 5d$ bridges of A', there are two bridges $\alpha_1, \alpha_2 \in A'$ at distance at least $\deg\beta - 1$ that intersect β. There are at least $\deg\beta - 4d$ bridges in A' that lie between α_1 and α_2 in the linear order $\prec_1$, at distance at least $2d$ from both α_1 and α_2: all these bridges must intersect β (see Fig. 1, right).

Partition the integers $\{1, 2, \ldots, |A'|\}$ into intervals $I_1 \cup I_2 \cup \ldots \cup I_s$, each of size between $d/3$ and $d/2$. Clearly, the number of intervals, s, satisfies $s \leq \frac{|A'|}{d/3} \leq \frac{n}{\varepsilon n/3} \leq \frac{3}{\varepsilon}$. For each $\beta \in B'$, $\deg\beta \geq 5d$, there are at least

$$\left\lfloor \frac{\deg\beta - 4d}{d/2} \right\rfloor - 1 \geq \left\lfloor \frac{2}{d}\deg\beta \right\rfloor - 9 > 0$$

intervals I such that β intersects every bridge of A whose label belongs to I. Taking (1) into account, the total number of such intervals I over all $\beta \in B'$ is at least

$$\frac{2}{d} \sum_{\beta \in B'} \deg\beta - 10|B'| \geq 14\varepsilon n - 10\varepsilon n = 4\varepsilon n.$$

Hence, by the pigeonhole principle, there is an interval I and a subset $B'' \subset B'$ with $|B''| \geq 4\varepsilon n/s \geq \varepsilon^2 n$ such that every element of B'' intersects every element of A' whose label belongs to I. Let $A'' \subset A'$ denote the set of all bridges whose labels belong to I. Obviously, we have $|A''| \geq d/3 = \varepsilon n/3 > \varepsilon^2 n$. $\square$

For the proof of Theorem 1, we also need to analyze the interaction between *convex* bridges, provided that their intersection graph is relatively dense. In the following lemma, the assumption of convexity is crucially important and cannot be replaced by vertical convexity.

For any compact convex bridge α between two vertical lines L_1 and L_2, let $s(\alpha)$ denote the segment connecting the points of $\alpha \cap L_1$ and $\alpha \cap L_2$ whose y-coordinates are *minimal.* The *lower curve* $\ell(\alpha)$ of α is the lower portion of the boundary of α between the two endpoints of $s(\alpha)$. Analogously, the *upper curve* $u(\alpha)$ of α is defined as the upper portion of the boundary of α connecting the points of $\alpha \cap L_1$ and $\alpha \cap L_2$ with *maximal* y-coordinates. Obviously, the lower (upper) curve of α is the graph of some convex (concave) function $f : [a, b] \to \mathbb{R}$.

Lemma 14 *Let A be a set of at most n convex bridges between two vertical lines L_1 and L_2. If there are at least εn^2 intersecting pairs in A, for some $\varepsilon > 0$, then the intersection graph of the bridges contains a bi-clique of size at least $\lfloor \varepsilon n/6 \rfloor$.*

Proof: Partition the intersecting pairs of A into five color classes as follows. Color an unordered pair $(\alpha, \beta) \in A \times A$ with color i ($1 \leq i \leq 5$), according to the first rule that applies to it (see Fig. 2). Use

color 1 if α and β intersect along L_1 or L_2 (that is, if $(\alpha \cap \beta) \cap (L_1 \cup L_2) \neq \emptyset$);
color 2 if the segments $s(\alpha)$ and $s(\beta)$ intersect;
color 3 if the lower curves $\ell(\alpha)$ and $\ell(\beta)$ intersect;
color 4 if the upper curves $u(\alpha)$ and $u(\beta)$ intersect;
color 5 if $\ell(\alpha)$ and $u(\beta)$ intersect or $u(\alpha)$ and $\ell(\beta)$ intersect.

We show that the intersection graph of A contains a bi-clique of size $\lfloor \varepsilon n/6 \rfloor$ in one of the color classes.

Case 1: At least $\varepsilon n^2/3$ pairs have color 1.

Suppose without loss of generality that at least $\varepsilon n^2/6$ pairs intersect along L_1. Consider the system of intervals obtained by intersecting the elements of A with L_1. If there is point of L_1 covered by at least $\varepsilon n/6$ intervals, then their intersection graph contains a *clique*, and hence a bi-clique, of size $\lfloor \varepsilon n/6 \rfloor$, so we are done. Otherwise, it is easy to see (and is well known) that the intersection graph of the intervals is $(\lfloor \varepsilon n/6 \rfloor) - 2)$-degenerate, that is, every subgraph contains a vertex of degree at most $(\lfloor \varepsilon n/6 \rfloor) - 2)$. Hence its number of edges is smaller than $(\lfloor \varepsilon n/6 \rfloor - 2)n < \varepsilon n^2/6$, contradicting our assumption.

Case 2: At least $\varepsilon n^2/6$ pairs have color 2.

Every pair in A that has color 2 is incomparable under the 2-dimensional partial order $\prec$, introduced at the beginning of this section. By Theorem 3, the incomparability graph of $(A, \prec)$ contains a bi-clique

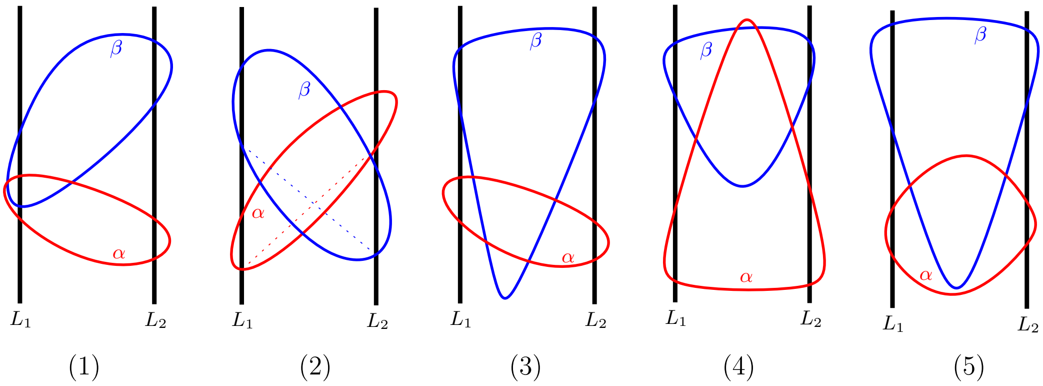

Figure 2: Pairs of intersecting convex bodies of color 1, 2, 3, 4, and 5.

whose size is at least the number of its edges divided by the number of its vertices. In our case, this means that the incomparability graph of $(A, \prec)$ contains a bi-clique of size at least $\varepsilon n/6$. Since every pair that is incomparable under $\prec$ must intersect, we are done.

Cases 3 and 4: At least $\varepsilon n^2/6$ pairs have color 3 (color 4).

For any $\alpha \in A$, let $\Gamma_3(\alpha)$ denote the set of all convex bridges $\beta \in A$ such that $\alpha \prec \beta$ and $\text{color}(\alpha, \beta) = 3$. It is easy to verify that any two elements $\beta, \gamma \in \Gamma_3(\alpha)$ intersect. Indeed, if β and γ were disjoint for some $\beta \prec \gamma$, then the segment $s(\alpha)$ would separate $\ell(\alpha)$ from γ, and hence from $\ell(\gamma)$, in the vertical strip, showing that $\gamma \notin \Gamma_3(\alpha)$, a contradiction.

If at least $\varepsilon n^2/6$ pairs have color 3, then there exists $\alpha_0 \in A$ for which $|\Gamma_3(\alpha_0)| \geq \varepsilon n/6$. The intersection graph of $\Gamma_3(\alpha_0)$ is a clique of size at least $\varepsilon n/6$. (An analogous argument applies if at least $\varepsilon n^2/6$ pairs have color 4.)

Case 5: At least $\varepsilon n^2/6$ pairs have color 5.

For any $\alpha \in A$, let $\Gamma_5(\alpha)$ denote the set of all $\beta \in A$ such that $\alpha \prec \beta$ and $u(\alpha) \cap \ell(\beta) \neq \emptyset$, but $\ell(\alpha) \cap \ell(\beta) = \emptyset$ and $u(\alpha) \cap u(\beta) = \emptyset$. It is easy to verify that now any two elements $\beta, \gamma \in \Gamma_5(\alpha)$ intersect. If at least $\varepsilon n^2/6$ pairs have color 5, then there exists $\alpha_0 \in A$ for which $|\Gamma_5(\alpha_0)| \geq \varepsilon n/6$. The intersection graph of $\Gamma_5(\alpha_0)$ is a clique of size at least $\varepsilon n/6$, completing the proof in this last case. $\square$

6 Proofs of Theorems 1 and 2

We are now in a position to prove Theorems 1 and 2.

First we prove Theorem 1, which states that the family of intersection graphs of finite systems of convex sets in the plane has the strong Erdős-Hajnal property. That is, we show that there exists a constant $c > 0$ such that the intersection graph G of any system of n convex sets in the plane contains a bi-clique of size at least cn, or the complement of G contains a bi-clique of this size. One can easily argue that it is sufficient to consider intersection graphs of systems S consisting of n convex *polygons*. We also assume without loss of generality that all the x-coordinates $\{\min_{p \in \alpha} x(p), \max_{q \in \alpha} x(q) : \alpha \in S\}$ are distinct.

If there is no vertical line that intersects at least $n/3$ elements of S, then pick a vertical line L not tangent to any polygon in S such that the number of elements of S lying entirely in the (open) half-plane to the left of L is precisely $\lfloor n/3 \rfloor$. Then there are at least $\lfloor n/3 \rfloor$ sets in the half-plane to the right of L, showing that $\overline{G}$, the complement of the intersection graph of S, contains a bi-clique of size $2\lfloor n/3 \rfloor$, and we are done.

Therefore, we can assume that there is a vertical line L that intersects $m \geq n/3$ elements of S, and from now on we will concentrate on the intersection graph G_m of these m elements. If the average degree of the vertices in G_m is less than $m/12^8$, then by Theorem 11 we can conclude that $\overline{G}_m$, and hence $\overline{G}$, contains a bi-clique of size at least $2m/12^5 > 2n/10^6$.

We are left with the case when $m \geq n/3$ elements of S intersect a vertical line L, and the average degree of the vertices in their intersection graph G_m is larger than $m/12^8$. This latter condition means that $|E(G_m)| > m^2/(2 \cdot 12^8) > m^2/10^9$. This case is resolved in the next Theorem 15, in which we do not even require that all sets be crossed by a vertical line.

Theorem 15 *For any system S of n convex sets in the plane with at least δn^2 intersecting pairs, the intersection graph of S contains a bi-clique of size at least $\lfloor \delta^2 n/600 \rfloor$.*

Proof: As before, we assume that all elements of S are convex polygons. On the boundary of each polygon in S, we fix a leftmost point and a rightmost point, and we assume that the x-coordinates of these $2n$ *extreme* points are all different. For any intersecting pair of polygons, choose a point that belongs to their intersection. Using vertical lines that do not pass through any of these special points, divide the plane into $\Delta := \lceil 10/\delta \rceil + 1$ strips (the leftmost and rightmost of which are just half-planes) such that each strip contains at most $\lceil 2n/\Delta \rceil \leq 1 + \delta n/5 < \delta n/4$ extreme points. (In the last inequality we assumed that $n > 20/\delta$. Indeed, the statement of the theorem is void if $n \leq 20/\delta$.)

In at least one of these strips, there will be at least $\delta n^2/\Delta \geq \delta^2 n^2/11$ special intersection points. Choose such a strip R. Since the leftmost and rightmost strips each contain at most $\binom{\delta n/4}{2} < \delta^2 n^2/32$ special points, R lies between two vertical lines L_1 and L_2. Clip in R each convex polygon in S that intersects the interior of R, and denote the resulting system of nonempty polygons by S'. Let A be the set of polygons in S' that form a bridge between L_1 and L_2, and let $B = S' \setminus A$. By the construction, B has at most as many elements as the number of extreme points in R. That is, we have $|B| \leq \delta n/4$.

Case 1: If there are at least $\delta^2 n^2/100$ intersecting pairs in A, then, by Lemma 14, the intersection graph of A contains a bi-clique of size $\lfloor \delta^2 n/600 \rfloor$.

Case 2: Suppose there are fewer than $\delta^2 n^2/100$ intersecting pairs in A and at least $\delta^2 n^2/11$ intersecting pairs in $A \cup B$. Now we can apply Lemma 13 to the system $S' = A \cup B$ with $\varepsilon = \delta/20$, to conclude that the intersection graph of $A \cup B$ contains a bi-clique of size $2\varepsilon^2 n = \delta^2 n/200$. $\square$

Our proof for Theorem 2 is analogous. The only difference is that, instead of Theorem 15, we complete the proof using the following assertion.

Theorem 16 *For any system S of n x-monotone curves in the plane with at least δn^2 intersecting pairs, the intersection graph of S contains a bi-clique of size at least $\lfloor c\frac{\delta^2}{\log 1/\delta} n/\log n \rfloor$, where $c > 0$ is an absolute constant.*

The proof of Theorem 16 is almost identical with the proof of Theorem 15, except that in Case 1 we have to use Theorem 4 instead of Lemma 14, as this latter statement heavily used the assumption that the bridges are *convex*. Note that the condition of x-monotonicity was crucial in the proof when we assumed that the portions of each curve clipped in a vertical strip is connected.

7 Concluding Remarks

Define the *edge density* of G as $2|E(G)|/n^2$, that is, as the average degree of G divided by $|V(G)|$. We have shown that the family of intersection graphs G of convex sets in the plane has the strong Erdős-Hajnal property. In particular, if the edge density of G is at least 12^{-8}, then the intersection graph contains a bi-clique of linear size (Theorem 15), otherwise its complement does so (Theorem 11). We show in Theorem 15 that if the edge density is δ, $0 < \delta \leq 1$, then the graph contains a bi-clique of size $\Omega(\delta^2 n)$. We do not know if the dependence on δ can be improved; the right bound might be $\Omega(\delta n)$. This problem can be restated as follows.

Problem 17 *Does every intersection graph G of convex sets in the plane with average degree d contain a bi-clique of size $\Omega(d)$?*

We do not know either if any statement analogous to Theorem 15 holds for the complements of the intersection graphs of convex sets.

Problem 18 *Does there exist a function $f : [0, 1) \to \mathbb{R}_{>0}$ such that if the intersection graph G of n convex sets in the plane has edge density at most δ for some $0 \leq \delta < 1$, then the complement of G contains a bi-clique of size at least $f(\delta)n$.*

Szemerédi's regularity lemma [14, 20] is an extremely powerful tool in studying structural properties of graphs whose edge densities are strictly separated from 0 and 1. In fact, this lemma played a crucial role in the discovery and in the first proof of Theorems 1 and 2, and perhaps it can also help in the solution of the above problems.

In a companion paper [13], we prove that for every $k \in \mathbb{N}$, the family of intersection graphs of sets of curves in the plane with no pair intersecting in more than k points also has the strong Erdős-Hajnal property. In the proofs presented in this paper, partial orders played an important role. If we give up x-monotonicity, then it is hard to introduce meaningful orderings on a set of curves, so most methods developed in this paper seem to break down.

Erdős and Szekeres [10] proved in 1935 that every sequence of $n^2 + 1$ distinct real numbers contains an increasing or decreasing subsequence of length $n+1$. This result quickly follows from Dilworth's theorem. A sequence of distinct real numbers naturally comes with a 2-dimensional partial order $\prec$, where $x_i \prec x_j$ if and only if $i < j$ and $x_i < x_j$. An increasing sequence corresponds to a chain in the partial order, a decreasing sequence corresponds to an antichain.

A bipartite analogue of the Erdős-Szekeres result is a simple consequence of the following result in [11] for 2-dimensional partial orders: For every sequence of n distinct real numbers, there are disjoint subsets A and B with $|A| = |B| = \lfloor \frac{n}{4} \rfloor$ such that the index of every element in A is larger than the index of every element of B; and either every element of A is larger than every element of B or every element of A is smaller than every element of B. Theorem 3 immediately implies the following.

Corollary 19 *Every sequence $x_1, \ldots, x_n$ of $n \geq 2t - 1$ real numbers such that $x_i < x_j$ holds for more than $2(t-1)n - \binom{2t-1}{2}$ pairs $i < j$ with $i, j \in [n]$, has two disjoint subsequences A and B of size $|A| = |B| = t$ such that every element of A is smaller than every element of B, and the index of every element of A is smaller than the index of every element of B.*

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