On the minimum degree of minimal Ramsey graphs for multiple colours

Jacob Fox[*]   Andrey Grinshpun[†]   Anita Liebenau[‡]   Yury Person[§]

Tibor Szabó[¶]

February 9, 2015

Abstract

A graph G is r-Ramsey for a graph H, denoted by $G \to (H)_r$, if every r-colouring of the edges of G contains a monochromatic copy of H. The graph G is called r-Ramsey-minimal for H if it is r-Ramsey for H but no proper subgraph of G possesses this property. Let $s_r(H)$ denote the smallest minimum degree of G over all graphs G that are r-Ramsey-minimal for H. The study of the parameter s_2 was initiated by Burr, Erdős, and Lovász in 1976 when they showed that for the clique $s_2(K_k) = (k-1)^2$. In this paper, we study the dependency of $s_r(K_k)$ on r and show that, under the condition that k is constant, $s_r(K_k) = r^2 \cdot \operatorname{polylog} r$. We also give an upper bound on $s_r(K_k)$ which is polynomial in both r and k, and we determine $s_r(K_3)$ up to a factor of $\log r$.

1 Introduction

A graph G is r-Ramsey for a graph H, denoted by $G \to (H)_r$, if every r-colouring of the edges of G contains a monochromatic copy of H. The fact that, for any number of colours r and every graph H, there exists a graph G such that $G \to (H)_r$ is a consequence of Ramsey's

theorem [19]. Many interesting questions arise when we consider graphs G which are minimal with respect to $G \to (H)_r$. A graph G is *r-Ramsey-minimal for H* (or *r-minimal for H*) if $G \to (H)_r$, but $G' \nrightarrow (H)_r$ for any proper subgraph $G' \subsetneq G$. Let $\mathcal{M}_r(H)$ denote the family of all graphs G that are r-Ramsey-minimal with respect to H. Ramsey's theorem implies that $\mathcal{M}_r(H)$ is non-empty for all integers r and all finite graphs H. However, for general H, it is still widely open to classify the graphs in $\mathcal{M}_r(H)$, or even to prove that these graphs have certain properties.

Of particular interest is $H = K_k$, the complete graph on k vertices, and a fundamental problem is to estimate various parameters of graphs $G \in \mathcal{M}_r(K_k)$, that is, of r-Ramsey-minimal graphs for the clique on k vertices. The best-studied such parameter is the Ramsey number $R_r(H)$, the smallest number of vertices of any graph in $\mathcal{M}_r(H)$. Estimating $R_r(K_k)$, or even $R_2(K_k)$, is one of the main open problems in Ramsey theory. Classical results of Erdős [13] and Erdős and Szekeres [14] show that $2^{k/2} \leqslant R_2(k) \leqslant 2^{2k}$. While there have been several improvements on these bounds (see for example [8] and [23]), the constant factors in the above exponents remain the same. For multiple colours, the gap between the bounds is larger. Even for the triangle K_3, the best known upper bound on the r-colour Ramsey number $R_r(K_3)$ is of order $2^{O(r \ln r)}$ [25], whereas, from the other side, $R_r(K_3) \geqslant 2^{\Omega(r)}$ is the best known lower bound (see [27] for the best known constant).

Other properties of $\mathcal{M}_r(K_k)$ have also been studied: Rödl and Siggers showed in [20] that, for all $k \geqslant 3$ and $r \geqslant 2$, there exists a constant $c = c(r, k) > 0$ such that, for n large enough, there are at least 2^{cn^2} non-isomorphic graphs G on at most n vertices that are r-Ramsey-minimal for the clique K_k. In particular, $\mathcal{M}_r(K_k)$ is infinite. Another well-studied parameter is the size Ramsey number $\hat{R}_r(H)$ of a graph H, which is the minimum number of edges of a graph in $\mathcal{M}_r(K_k)$.

Interestingly, some extremal parameters of graphs in $\mathcal{M}_r(K_k)$ could be determined exactly when the number of colours is two. In this paper, we consider the minimal minimum degree of r-Ramsey-minimal graphs $s_r(H)$, defined by

$$s_r(H) := \min_{G \in \mathcal{M}_r(H)} \delta(G)$$

where $\delta(G)$ denotes the minimum degree of G.

It is rather simple to see that, for any graph H,

$$r(\delta(H) - 1) < s_r(H) < R_r(H). \tag{1}$$

Indeed, for $r = 2$, the proof of the lower bound is included in [16]; it generalises easily to more colours. We include a similar argument at the beginning of Section 3. In [6], Burr, Erdős, and Lovász showed that, rather surprisingly, the simple upper bound above is far from optimal when $r = 2$, namely $s_2(K_k) = (k-1)^2$.

In this paper, we study the behaviour of $s_r(K_k)$ as a function of r and k. We mainly study $s_r(K_k)$ as a function of r with k fixed. In particular, we determine $s_r(K_3)$ up to a logarithmic factor.

Theorem 1.1. *There exist constants $c, C > 0$ such that for all $r \geqslant 2$, we have*

$$cr^2 \ln r \leqslant s_r(K_3) \leqslant Cr^2 \ln^2 r.$$

One can show that $s_r(K_k) \geqslant s_{r-1}(K_k)$ (this follows from a stronger statement, cf. Theorem 1.5 and Proposition 3.2). However, it is not clear that $s_r(K_k) \geqslant s_r(K_{k-1})$. Therefore, the lower bound on $s_r(K_3)$ does not necessarily imply a similar lower bound on $s_r(K_k)$. We can in fact only prove a super-quadratic lower bound on $s_r(K_k)$ that is slightly weaker.

Theorem 1.2. *For all $k \geqslant 4$ there exist constants $c = c(k), C = C(k) > 0$ such that, for all $r \geqslant 3$,*

$$c\,r^2 \frac{\ln r}{\ln \ln r} \leqslant s_r(K_k) \leqslant C\,r^2 (\ln r)^{8(k-1)^2}.$$

The proof of the upper bounds in Theorems 1.1 and 1.2 are of asymptotic nature and require r to be rather large. Moreover, the exponent of the $(\ln r)$-factor in the latter upper bound depends on the size of the clique. Therefore, we also prove an upper bound on $s_r(K_k)$ which is polynomial both in r and in k and is applicable for small values of r and k.

Theorem 1.3. *For $k \geqslant 3$, $r \geqslant 3$, $s_r(K_k) \leqslant 8(k-1)^6 r^3$.*

Tools. We give an overview of the tools we use to prove bounds on $s_r(K_k)$. The first step will be to reduce finding $s_r(K_k)$ to a simpler problem. We call a sequence of pairwise edge-disjoint graphs $G_1, \ldots, G_r$ on the same vertex set V a *colour pattern* on V. For a graph H, a colour pattern $G_1, \ldots, G_r$ is called H-*free* if none of the G_i contains H as a subgraph. A graph with coloured vertices and edges is called *strongly monochromatic* if all its vertices and edges have the same colour.

Definition 1.4. *The r-colour k-clique packing number, $P_r(k)$, is the smallest integer n such that there exists a K_{k+1}-free colour pattern $G_1, \ldots, G_r$ on an n-element vertex set V with the property that any $[r]$-colouring of V contains a strongly monochromatic K_k.*

While Burr, Erdős, and Lovász [6] do not explicitly define $P_2(k)$ in their proof of $s_2(K_k) = (k-1)^2$, they do essentially show that $s_2(K_k) = P_2(k-1)$ and it is then not hard to see that $P_2(k-1) = (k-1)^2$. Here we generalise their result to an arbitrary number r of colours.

Theorem 1.5. *For all integers $r, k \geqslant 2$ we have $s_r(K_{k+1}) = P_r(k)$.*

The lower bound $s_r(K_{k+1}) \geqslant P_r(k)$ is not difficult to derive from the definitions. The upper bound $s_2(K_{k+1}) \leqslant P_2(k)$ follows from a powerful theorem of [6]. We use later generalisations of this theorem by Burr, Nešetřil, and Rödl [7] and, recently in 2008, by Rödl and Siggers [20] to derive $s_r(K_{k+1}) \leqslant P_r(k)$ for arbitrary $r \geqslant 2$.

The problem then becomes to obtain bounds on $P_r(k)$. We will see that $P_r(k)$ relates closely to the so-called *Erdős-Rogers function*, which was first studied by Erdős and Rogers [12] in 1962. We will be particularly concerned with the special case of the Erdős-Rogers function, denoted by $f_{k,k+1}(n)$, which is defined to be the largest integer α so that in any K_{k+1}-free graph on n vertices, there must be a vertex-set of size α that contains no K_k. For our bounds, we will rely heavily on the modern analysis of $f_{k,k+1}$ found in [9, 10, 11, 21]. In Section 3, we will see that essentially $P_r(k) = \Omega\left(r(f_{k,k+1}(r))^2\right)$, so lower bounds on $f_{k,k+1}$ directly translate

to lower bounds on $P_r(k)$. In Section 4, we obtain upper bounds on $P_r(k)$ by packing r graphs, each giving good upper bounds on $f_{k,k+1}$, into the same vertex set.

Organisation. In the next section, we prove that $s_r(K_{k+1}) = P_r(k)$. In Section 3, we prove the lower bounds on $P_r(k)$ in Theorem 1.1 and Theorem 1.2. In Section 4, we prove the upper bounds in Theorem 1.1, Theorem 1.2, and Theorem 1.3. We close this paper with some concluding remarks.

2   Passing to $P_r(k)$

In this section we conclude Theorem 1.5 from Lemmas 2.1 and 2.3.

Lemma 2.1. *For all $r, k \geqslant 1$, we have $s_r(K_{k+1}) \geqslant P_r(k)$.*

Proof. Let G be an r-Ramsey-minimal graph for K_{k+1} with a vertex v of degree $s_r(K_{k+1})$. Let $\chi : E(G - v) \to [r]$ be an r-colouring of $G - v$ without a monochromatic K_{k+1}; such a colouring exists by the minimality of G. Let $G_1, \ldots, G_r \subseteq G[N(v)]$ be the pairwise edge-disjoint subgraphs of the r colours within the neighbourhood $N(v)$ of v; they form a K_{k+1}-free colour pattern on $N(v)$. We show that any vertex-colouring of $G[N(v)]$ must contain a strongly monochromatic k-clique and hence, by the definition of $P_r(k)$, the number of vertices $|N(v)| = s_r(K_{k+1})$ must be at least $P_r(k)$. Indeed, given any vertex-colouring of $N(v)$ we may define an extension of χ to the edges incident to v by colouring an edge vu with the colour of the vertex $u \in N(v)$. Since G is r-Ramsey for K_{k+1}, this extension of χ contains a monochromatic $(k+1)$-clique H. Moreover, H must contain v (as χ was free of monochromatic K_{k+1}). By the definition of the extension of χ, the vertices of H in $N(v)$ form a strongly monochromatic K_k in $G[N(v)]$. $\qquad\square$

In order to show $s_r(K_{k+1}) \leqslant P_r(k)$, we first prove a theorem that guarantees, for any integer $r \geqslant 2$ and graph H which is 3-connected or a triangle, a fixed colour pattern on a given induced subgraph of some graph G which is not r-Ramsey for H, in *any* monochromatic H-free r-colouring of G. A similar theorem was proved for $H = K_k$ and for $r = 2$ in [6], where they use it to show $s_2(K_{k+1}) \leqslant P_2(k)$. The tools used to prove this were generalised to any 3-connected graph H in [7], and, more recently, to any number of colours and any graph H which is 3-connected or a triangle [20].

Theorem 2.2. *Let H be any 3-connected graph or $H = K_3$ and let $G_1, \ldots, G_r$ be an H-free colour pattern. Then there is a graph G with an induced copy of the edge-disjoint union $G_1 \cup \cdots \cup G_r$ so that $G \nrightarrow (H)_r$ and in any monochromatic H-free r-colouring of $E(G)$ each G_i is monochromatic and no two distinct G_i and G_j are monochromatic of the same colour.*

Proof. We use the idea of signal sender graphs which was first introduced by Burr, Erdős and Lovász [6]. Let $r \geqslant 2$ and $d \geqslant 0$ be integers and H be a graph. A *negative (positive) signal sender* $S = S^-(r, H, d)$ $(S = S^+(r, H, d))$ is a graph S with two distinguished edges $e, f \in E(S)$ of distance at least d, such that

4

(a) $S \nrightarrow (H)_r$, and

(b) in every r-colouring of $E(S)$ without a monochromatic copy of H, the edges e and f have different (the same) colours.

We call e and f the *signal edges* of S.

Burr, Erdős and Lovász [6] showed that positive and negative signal senders exist for arbitrary d in the special case when the number of colours is two, and H is a clique on at least three vertices. Later, Burr, Nešetřil and Rödl [7] extended these results to arbitrary 3-connected H. Finally, Rödl and Siggers [20] constructed positive and negative signal senders $S^-(r, H, d)$ and $S^+(r, H, d)$ for any $r \geqslant 3$, $d \geqslant 0$ as long as H is 3-connected or $H = K_3$.

Let H be a graph that is either 3-connected or $H = K_3$ and let $G_1, \ldots, G_r$ be an H-free colour pattern on vertex set V. We construct our graph G using the signal senders of Rödl and Siggers. We first take the graph on V which is the edge-disjoint union of the edge sets of the graphs G_i and add r isolated edges $e_1, \ldots, e_r$ disjoint from V. Then for every i and every edge $f \in E(G_i)$ we add a copy of $S^+(r, H, |V(H)|)$, such that f and e_i are the two signal edges and the sender graph is otherwise disjoint from the rest of the construction. Finally, for every pair of edges e_i, e_j, we add a copy of $S^-(r, H, |V(H)|)$, such that e_i and e_j are the two signal edges and the sender graph is otherwise disjoint from the rest of the construction.

By the properties of positive and negative signal senders, in any r-colouring of G without a monochromatic copy of H, each G_i must be monochromatic and no two G_i, G_j may be monochromatic in the same colour.

Now we need only to show that there exists an r-colouring of G with no monochromatic H. For this, we first colour each G_i with colour i. Then, we extend this colouring to a colouring of each signal sender so that each signal sender contains no monochromatic copy of H. This is possible since each positive (negative) signal sender has a colouring without a monochromatic copy of H in which the signal edges have the same (different) colours. Let us consider a copy of H in G. We will see that H is contained either within $G_1 \cup \cdots \cup G_r$ or within one of the signal senders and hence it is not monochromatic. If this was not the case, then there would be a vertex v_1 of H that is not in any of the signal edges, that is, $v_1 \in V(S)$ for some signal sender S but not contained in any of the two signal edges of S. Since H is not entirely in S, there must be a vertex $v_2 \in V(H) \setminus V(S)$. This immediately implies that $H \neq K_3$, since v_1 and v_2 are not adjacent. Since H is 3-connected there are three internally disjoint v_1, v_2-paths in H. These paths can leave S only through one of its two signal edges. Hence there is a path of H in S between the two signal edges. This is a contradiction because the distance of the two signal edges in S is at least $|V(H)|$. $\qquad \square$

Theorem 2.2 allows us to finish the proof of Theorem 1.5.

Theorem 2.3. $s_r(K_{k+1}) \leqslant P_r(k)$.

Proof. Let a K_{k+1}-free colour pattern $G_1, \ldots, G_r$ be given on vertex set V with $|V| = P_r(k)$, so that any $[r]$-colouring of V contains a strongly monochromatic K_k. Take G as in Theorem

2.2 with $H = K_{k+1}$, and define G' to be G with a new vertex v which is incident only to V. We claim that $G' \to K_{k+1}$, that is for any r-colouring χ of G' we find a monochromatic K_{k+1}. If already the restriction of χ to $V(G)$ contains a monochromatic K_{k+1} then we are done. Otherwise, by Theorem 2.2, we have that, after potentially permuting the colours, each subgraph $G_i \subseteq G[V]$ is monochromatic in colour i. We define a colouring of V by colouring $u \in V$ with $\chi(uv)$. Then, by the choice of $G_1, \dots, G_r$, there is a strongly monochromatic clique in V. This clique along with vertex v forms a monochromatic K_{k+1} in the colouring χ.

So $G' \to K_{k+1}$. Now observe that any r-Ramsey-minimal subgraph of G' must contain the vertex v, since $G' - v = G$ is not r-Ramsey for K_{k+1} by Theorem 2.2. Hence for the minimum degree of any r-Ramsey-minimal subgraph $G'' \subseteq G'$ we have that $s_r(K_{k+1}) \leqslant \delta(G'') \leqslant \deg_{G''}(v) \leqslant \deg_{G'}(v) = P_r(k)$. $\qquad\square$

3 Lower bounds on $P_r(k)$

First, we prove a simple linear lower bound on $P_r(k)$. This simple estimate will later be used to obtain a super-quadratic lower bound.

Lemma 3.1. *For all $r \geqslant 2$ and $k \geqslant 3$, we have $P_r(k) > (k-1)r$.*

Proof. We will show that for any given colour pattern $G_1, \dots, G_r$ on vertex set V, $|V| \leqslant (k-1)r$, there is a vertex-colouring of V without a strongly monochromatic K_k and hence, $P_r(k) > (k-1)r$. Observe that every vertex $v \in V$ has degree at most $k-2$ in at least one of the colour classes, say $G_{i(v)}$. Colouring vertex v with colour $i(v)$ ensures that v is not contained in any strongly monochromatic K_k, as its degree in $G_{i(v)}$ is too low. Hence, as promised, this vertex-colouring of V produces no strongly monochromatic K_k. $\qquad\square$

For a graph F, the *k-independence number* $\alpha_k(F)$ is the largest cardinality of a subset $I \subseteq V(F)$ without a K_k. For $k = 2$, this is the usual independence number $\alpha(F)$. Recall that the Erdős-Rogers function $f_{k,k+1}(n)$ is defined to be the minimum value of $\alpha_k(F)$ over all K_{k+1}-free graphs F on n vertices.

The following proposition provides the recursion for our lower bound.

Proposition 3.2. *For all $r, k \geqslant 2$ we have that $P_r(k)$ satisfies the following inequality:*

$$P_r(k) \geqslant P_{r-1}(k) + f_{k,k+1}(P_r(k))$$

Proof. Take $G_1, \dots, G_r$ to be a K_{k+1}-free colour pattern on vertex set V, $|V| = P_r(k)$, so that any r-colouring of the vertices contains a strongly monochromatic K_k. Let $I \subseteq V$ be a k-independent set of size $\alpha_k(G_r)$ in the graph G_r. We claim that the K_{k+1}-free colour pattern $G_1, \dots, G_{r-1}$ restricted to the vertex set $V \setminus I$ has the property that any $[r-1]$-colouring $c : V \setminus I \to [r-1]$ contains a strongly monochromatic K_k. Indeed, the extension of c to V which colours the vertices in I with colour r must contain a strongly monochromatic K_k and

this must be inside $V \setminus I$, since I does not contain K_k at all. Hence $|V \setminus I| \geqslant P_{r-1}(k)$ and then, since G_r is a K_{k+1}-free graph on $P_r(k)$ vertices, we have that

$$P_r(k) = |V \setminus I| + |I| \geqslant P_{r-1}(k) + \alpha_k(G_r) \geqslant P_{r-1}(k) + f_{k,k+1}(P_r(k)). \qquad \square$$

Therefore, we are interested in good lower bounds on the Erdős-Rogers function $f_{k,k+1}(n)$. It is easy to see that every K_{k+1}-free graph F on n vertices contains a K_k-free set of size at least $\lfloor \sqrt{n} \rfloor$. If there exists a vertex v of degree at least $\lfloor \sqrt{n} \rfloor$, then $N(v)$ is a K_k-free set of size at least $\lfloor \sqrt{n} \rfloor$. Otherwise, $\Delta(F) \leqslant \lfloor \sqrt{n} \rfloor - 1$ and we can use the well-known fact that $\alpha(F) \geqslant n/(\Delta(F) + 1)$ (cf. [2]) to deduce that $\alpha_k(F) \geqslant \alpha(F) \geqslant \lfloor \sqrt{n} \rfloor$. Therefore, $f_{k,k+1}(n) \geqslant \lfloor \sqrt{n} \rfloor$.

A result of Shearer [21] implies that $f_{2,3}(n) \geqslant (1 - o(1)) \sqrt{(n \ln n)/2}$, which is the best known lower bound on $f_{2,3}(n)$. Bollobás and Hind [5] proved that $f_{3,4}(n) \geqslant \sqrt{2n}$. This lower bound was subsequently improved by Krivelevich [18]. Recently, Dudek and Mubayi [9] showed that this result can be strengthened to

$$f_{k,k+1}(n) = \Omega\left(\sqrt{\frac{n \log n}{\log \log n}}\right)$$

by using a result of Shearer [22].

Proof of the lower bounds in Theorem 1.1 and 1.2. Let k be fixed and and for brevity let us write $P_r := P_r(k)$. Let $f_{k,k+1}(n) \geqslant g(n)\sqrt{n}$ for $n \geqslant n_0$, where $g(n) = g_k(n)$ is a non-decreasing function such that $\frac{Cg^2(n-1)}{n} > g^2(n) - g^2(n-1) > 0$ for $n \geqslant n_0$ with some constant $C = C(k)$. Note that one can take $g_2(n) = \frac{1}{2}\sqrt{\ln n}$ by [21] and for $k \geqslant 3$ one can take $g_k(n) = c\sqrt{\frac{\ln n}{\ln \ln n}}$ with some constant $c = c(k)$ by [9].

We show that there exists a constant $c' = c'(k)$ such that for $r \geqslant n_0 + 1$,

$$P_r \geqslant c'(rg(r))^2,$$

which then implies the lower bounds in Theorems 1.1 and 1.2.

We prove this statement by induction on r. For $r = n_0 + 1$ this is true provided c' is chosen small enough. For $r > n_0 + 1$, by Proposition 3.2 and since $f_{k,k+1}$ is non-decreasing, we have that

$$P_r \geqslant P_{r-1} + f_{k,k+1}(P_{r-1}) \geqslant P_{r-1} + \sqrt{P_{r-1}}g(P_{r-1}).$$

Using the induction hypothesis, Lemma 3.1 and that g is non-decreasing for $r - 1 \geqslant n_0$, we obtain

$$P_r \geqslant c'((r-1)g(r-1))^2 + \sqrt{c'}(r-1)g(r-1)g(r-1)$$

$$\geqslant c'(rg(r))^2 + r(g(r-1))^2 \left(\sqrt{c'} - 2c' - c'r\left(\frac{(g(r))^2}{(g(r-1))^2} - 1\right) - \frac{\sqrt{c'}}{r}\right).$$

By our assumption on g the last term is positive, provided c' is small enough. $\qquad \square$

4   Packing (n, r, k)-critical graphs

In this section we prove the upper bounds in Theorems 1.1, 1.2 and 1.3. Our task is to derive upper bounds for $P_r(k)$, that is we want to find K_{k+1}-free colour patterns such that every r-colouring of the vertices produces a strongly monochromatic K_k. Let us first motivate the idea behind our proofs. Given a colour pattern $G_1, \dots, G_r$ on an n-element vertex set V and any $[r]$-colouring of V, at least one of the colours, say i, occurs n/r times. If every set of at least n/r vertices in G_i contains a K_k, then we must have a strongly monochromatic clique in colour i. This motivates the following definition: we call a graph F on n vertices (n, r, k)-critical if $K_{k+1} \nsubseteq F$ and $\alpha_k(F) < n/r$. We have thus obtained the following lemma.

Lemma 4.1. *If there exists a colour pattern $G_1, \dots, G_r$ where each G_i is (n, r, k)-critical, then $P_r(k) \leqslant n$.*

For the rest of this section, we will focus on packing r edge-disjoint (n, r, k)-critical graphs into the same n-element vertex set, such that n is as small as possible.

In order to produce at least one (n, r, k)-critical graph, let us recall the Erdős-Rogers function, defined as $f_{k, k+1}(n) = \min\{\alpha_k(F)\}$, where the minimum is taken over all K_{k+1}-free graphs F on n vertices. By definition, we have for all $u \in \mathbb{R}$ that

$$f_{k, k+1}(n) < u \quad \Longleftrightarrow \quad \text{there exists an } (n, n/u, k)\text{-critical graph.} \tag{2}$$

So the question whether at least one (n, r, k)-critical graph exists on n vertices is equivalent to the question whether $f_{k, k+1}(n) < n/r$.

When $k = 2$, an $(n, r, 2)$-critical graph is precisely an n-vertex triangle-free graph with independence number less than n/r. Hence an $(n, r, 2)$-critical graph exists if and only if $n < R(3, \lceil n/r \rceil)$. It is known that $R(3, k) = \Theta\left(k^2 / \ln k\right)$ where the upper bound was first shown by Ajtai, Komlós and Szemerédi [1] and the matching lower bound was first established by Kim [17]. Therefore, if G is an $(n, r, 2)$-critical graph, then $n \geqslant c \cdot r^2 \ln r$ for some constant $c > 0$, and $(n, r, 2)$-critical graphs do exist for $n = C \cdot r^2 \ln r$ for some constant $C > 0$. For our purpose, however, we need to pack r many $(n, r, 2)$-critical graphs in an edge-disjoint fashion into n vertices. The next lemma states that we can do so at the expense of a factor of $\ln r$.

Lemma 4.2. *Let r be an integer. Then there exists a colour pattern $G_1, \dots, G_r$ on vertex set $[n]$, where $n = O(r^2 \ln^2 r)$, such that each G_i is $(n, r, 2)$-critical.*

Lemma 4.2 together with Lemma 4.1 and Theorem 1.5 complete the proof of Theorem 1.1.

For fixed $k \geqslant 3$, Dudek, Retter, and Rödl [10] recently showed that $f_{k, k+1}(n) = O\left((\ln n)^{4k^2} \sqrt{n}\right)$. That is, they constructed a K_{k+1}-free graph F on n vertices (where n is large enough) such that every subset of $c(\ln n)^{4k^2} \sqrt{n}$ vertices contains a K_k. This is an (n, r, k)-critical graph F with $n = c^2\left((2 + o(1)) \ln r\right)^{8k^2} r^2$. Again, we would like to pack r of those graphs into K_n. But rather than taking a fixed (n, r, k)-critical graph F and pack it into K_n, we construct r (edge-disjoint) (n, r, k)-critical graphs $G_1, \dots, G_r$ simultaneously as subgraphs of K_n. As it

turns out, this simultaneous construction is only little harder than the construction itself in [10]; we prove it by black-boxing theorems from [10].

Lemma 4.3. *For all integers $k \geqslant 3$ there exist a constant $C = C(k) > 0$ and $r_0 \in \mathbb{N}$ such that, for all $r \geqslant r_0$, the following holds. There exists a colour pattern $G_1, \ldots, G_r$ on vertex set $[n]$, where $n \leqslant C (\ln r)^{8k^2} r^2$, such that each G_i is (n, r, k)-critical.*

Lemma 4.3 together with Lemma 4.1 and Theorem 1.5 complete the proof of Theorem 1.2.

For the upper bound in Theorem 1.3, we are motivated by graphs constructed by Dudek and Rödl in [11]. The graph F on n vertices constructed in [11] is (n, r, k)-critical with $n = O(k^6 r^3)$. Here it is not as clear to just refer to lemmas from [11] in order to do a "simultaneous" construction. So we will start the construction from scratch and provide all the details needed.

Lemma 4.4. *Let $k, r \geqslant 3$. Then there exists a colour pattern $G_1, \ldots, G_r$ on vertex set $[n]$, where $n \leqslant 8k^6 r^3$, such that each G_i is (n, r, k)-critical.*

Lemma 4.4 together with Lemma 4.1 and Theorem 1.5 imply Theorem 1.2.

4.1 Proofs of the Lemmas

In the rest of this section we prove Lemmas 4.2, 4.3, and 4.4, each concerned with packing (edge-disjointly) r graphs $G_1, \ldots, G_r$ which are all (n, r, k)-critical.

Packing many K_3-free graphs with small independence number.
Here, we prove Lemma 4.2. To that end, we will show the existence of a graph F on $n := Cr^2 \ln^2 r$ vertices, where $C = 1000$, which can be written as a union of edge-disjoint graphs $G_1, \ldots, G_r$ which are all K_3-free and without independent sets of size n/r. We will find the graphs G_i successively as subgraphs of K_n using the following.

Lemma 4.5 (Lovász Local Lemma, see, e.g., [2, Lemma 5.1.1]). *Let $A_1, A_2, \ldots, A_n$ be events in an arbitrary probability space. A directed graph $D = (V, E)$ on the set of vertices $V = \{1, \ldots, n\}$ is called a dependency digraph for the events $A_1, A_2, \ldots, A_n$ if for each i, $1 \leqslant i \leqslant n$, the event A_i is mutually independent of all the events $\{A_j : (i, j) \notin E\}$. Suppose that $D = (V, E)$ is a dependency digraph for the above events and suppose there are real numbers $x_1, \ldots, x_n$ such that $0 \leqslant x_i < 1$ and $\Pr(A_i) \leqslant x_i \prod_{(i,j) \in E} (1 - x_j)$ for all $1 \leqslant i \leqslant n$. Then*

$$\Pr\left(\bigwedge_{i=1}^{n} \overline{A_i}\right) \geqslant \prod_{i=1}^{n} (1 - x_i).$$

In particular, with positive probability no event A_i holds.

Given r, set $m := n/r = Cr \ln^2 r$ and $q := \binom{m}{2}/(2r)$. For a graph H on n vertices, we define $e_{\min}(m, H)$ ($e_{\max}(m, H)$) to be the smallest (largest) number of edges that appear in any subset $S \subseteq V(H)$ of size $|S| = m$. The following lemma is the crucial step to find the graphs G_i.

Lemma 4.6. *Let $H = (V, E)$ be a graph on n vertices, where $n \geqslant n_0$ is large enough, and assume $e_{\min}(m, H) \geqslant \binom{m}{2}/2$. Then there is a subgraph $H' \subseteq H$ on the same vertex set such that $H' = (V, E')$ is triangle-free, has no independent set on m vertices, and $e_{\max}(m, H') \leqslant q$.*

Proof. Let $c_1 = 1/4$ and $c_2 = 1/20$. Choose H' by including each edge of H independently with probability $p := c_1 n^{-1/2}$. For a subset $S \subseteq V$, let $e(S)$ and $e'(S)$ denote the number of edges in $H[S]$ and $H'[S]$, respectively. It suffices to show that H' is triangle-free, $e_{\min}(m, H') \geqslant 1$, and $e_{\max}(m, H') \leqslant q$ with positive probability. To that end, we want to apply the Lovász Local Lemma, and, therefore, we define the set of bad events in the natural way. Namely, for every $S \in \binom{V}{3}$ that forms a triangle in H, we set T_S to be the event that $H'[S]$ is a triangle as well. Clearly, the probability of such an event is $p_T := p^3$. Further, for every $S \in \binom{V}{m}$, we set I_S to be the event that either S is an independent set in H' or satisfies $e'(S) > q$. Then,

$$
\begin{aligned}
\mathbb{P}(I_S) &\leqslant \mathbb{P}(e'(S) = 0) + \mathbb{P}(e'(S) \geqslant q) \\
&\leqslant (1-p)^{e(S)} + \binom{e(S)}{q} p^q \\
&\leqslant (1-p)^{\binom{m}{2}/2} + \left(\frac{\binom{m}{2} ep}{q} \right)^q \\
&= (1-p)^{\binom{m}{2}/2} + (2epr)^q.
\end{aligned}
$$

Note that $(1-p)^{\binom{m}{2}/2} = \exp\left[-p\binom{m}{2}/2 \, (1+o(1)) \right] = e^{-pqr(1+o(1))}$ and $(2epr)^q = o(e^{-pqr(1+o(1))})$, since $pr \to 0$, so that for n large enough

$$
\mathbb{P}(I_S) \leqslant 2(1-p)^{\binom{m}{2}/2} =: p_I.
$$

Let $\mathcal{E}$ be the collection of bad events. That is, $\mathcal{E} = \{T_S : H[S] \cong K_3\} \cup \{I_S : S \in \binom{V}{m}\}$. In the auxiliary dependency graph D, we connect two of the events $A_S, A_{S'} \in \mathcal{E}$ if $|S \cap S'| \geqslant 2$. Then $A_S \in \mathcal{E}$ is mutually independent from the family of all $A_{S'}$ for which $\{A_S, A_{S'}\}$ is not an edge in this dependency graph. To apply the Lovász Local Lemma, we now bound the degrees in D. We denote by $N(E)$ the neighbours in the dependency graph D of the event E. If $|S| = 3$ we have

$$
\begin{aligned}
\left| N(T_S) \cap \{T_{S'} : |S'| = 3\} \right| &\leqslant 3n, \qquad \text{and} \\
\left| N(T_S) \cap \{I_{S'} : |S'| = m\} \right| &\leqslant \binom{n}{m}.
\end{aligned}
$$

If $|S| = m$ we have

$$
\begin{aligned}
\left| N(I_S) \cap \{T_{S'} : |S'| = 3\} \right| &\leqslant \binom{m}{2}(n-2) < \binom{m}{2} n, \qquad \text{and} \\
\left| N(I_S) \cap \{I_{S'} : |S'| = m\} \right| &\leqslant \binom{n}{m}.
\end{aligned}
$$

Therefore, by Lemma 4.5, if there exist real numbers $x, y \in [0, 1)$ such that

$$p_T \leqslant x(1-x)^{3n}(1-y)^{\binom{n}{m}} \tag{3}$$

$$p_I \leqslant y(1-x)^{\binom{m}{2}n}(1-y)^{\binom{n}{m}}, \tag{4}$$

then there exists a graph H' such that none of the events in $\mathcal{E}$ occurs. We show that these two conditions are fulfilled for $x = c_2 n^{-3/2}$ and $y = \binom{n}{m}^{-1}$. First note that, for n large enough,

$$x(1-x)^{3n}(1-y)^{\binom{n}{m}} = c_2 n^{-3/2} e^{-1}(1 + o(1)) > p^3,$$

so Inequality (3) holds. Now, (4) is equivalent to

$$2^{2/\binom{m}{2}}(1-p) \leqslant y^{2/\binom{m}{2}}(1-x)^{2n}(1-y)^{2\binom{n}{m}/\binom{m}{2}}.$$

We use $1 - p \leqslant e^{-p}$ and $1 - z \geqslant e^{-z-z^2}$ for $z \leqslant 0.6$ to claim (4) holds if

$$\exp\left[\frac{2\ln 2}{\binom{m}{2}} - p\right] \leqslant \exp\left[\frac{2\ln y}{\binom{m}{2}} - 2n(x + x^2) - \frac{2\binom{n}{m}}{\binom{m}{2}}(y + y^2)\right].$$

Now, $\frac{2\ln y}{\binom{m}{2}} \geqslant -\frac{4}{\sqrt{C}} n^{-1/2}(1 + o(1))$ and $1/m^2 = o\left(n^{-1/2}\right)$. So (4) holds if

$$\exp\left[-c_1 n^{-1/2}(1 + o(1))\right] \leqslant \exp\left[-(4/\sqrt{C} + 2c_2)n^{-1/2}(1 + o(1))\right],$$

which is satisfied by choice of C, c_1, c_2. Applying Lemma 4.5 yields the existence of a subgraph H' such that none of the events in $\mathcal{E}$ hold, i.e. H' has the desired properties. $\qquad\square$

Proof of Lemma 4.2. Let r large enough be given, and set $m := n/r = Cr\ln^2 r$ and $q := \binom{m}{2}/(2r)$ as before. Define $H_1 := K_n$. We choose our graphs inductively as subgraphs of H_1; given H_i for $i \leqslant r$ such that $e_{\min}(m, H_i) \geqslant \binom{m}{2} - (i-1)q$, we have since $i \leqslant r$ that

$$e_{\min}(m, H_i) > \binom{m}{2} - rq = \frac{1}{2}\binom{m}{2},$$

so, by Lemma 4.6, we may find G_i a subgraph of H_i with $e_{\max}(m, G_i) \leqslant q$ such that G_i is triangle-free and has no independent set on n/r vertices. Then take $H_{i+1} = H_i - G_i$. The graph H_{i+1} will be edge-disjoint from G_i (and, inductively, from $G_1, \ldots, G_{i-1}$), and

$$e_{\min}(m, H_{i+1}) \geqslant e_{\min}(m, H_i) - e_{\max}(m, G_i) \geqslant \binom{m}{2} - (i-1)q - q = \binom{m}{2} - iq,$$

as desired. $\qquad\square$

An upper bound tight up to a polylogarithmic factor in r

Here, we prove Lemma 4.3. We will rely heavily on the graphs constructed in [10] and use its construction as a black box.

Proof of Lemma 4.3. Fix $k \geqslant 3$ and let r be large enough. We need to construct r graphs on $n = O(r^2 (\log r)^{8k^2})$ vertices that are K_{k+1}-free, but every subset of size n/r contains a K_k. Let q be the largest prime power such that

$$q \leqslant 128k(2\log r)^{4k^2}r.$$

Then by Bertrand's postulate, $q \geqslant 64k(2\log r)^{4k^2}r$, and therefore, $q \geqslant 64k(\log q)^{4k^2}r$ since r is large enough compared to k. Consider the affine plane of order q. It has $n := q^2$ points and $q^2 + q$ lines such that any two points lie on a unique line, every line contains q points, and every point lies on $q + 1$ lines. It is a well-known fact that affine planes exist whenever q is a prime power. We call two lines L and L' in the affine plane *parallel* if $L \cap L' = \emptyset$. In the affine plane of order q, there exist $q + 1$ sets of q pairwise disjoint lines. Let $(V, \mathcal{L})$ be a hypergraph where the vertex set V is the point set of the affine plane of order q, and the hyperedges are lines of the affine plane, with one set of parallel lines removed. Then $(V, \mathcal{L})$ is a q-uniform hypergraph on q^2 vertices such that any two hyperedges meet in at most one vertex.

In [10], Dudek et al. consider a random subhypergraph $(V, \mathcal{L}')$ of $(V, \mathcal{L})$ and show that they can embed the required graph G "along the hyperedges" of $(V, \mathcal{L}')$. For our purposes, let us call a hypergraph $(V, \mathcal{H})$ *good* if there exists a graph G on vertex set V such that

(i) $K_{k+1} \nsubseteq G$,

(ii) every subset of size $64k(\log q)^{4k^2}q$ of V contains a K_k in G, and

(iii) any edge of G lies inside a hyperedge of $\mathcal{H}$, i.e. for every $e \in E(G)$ there is some $h \in \mathcal{H}$ such that $e \subseteq h$.

Clearly, by (i) and (ii) any such graph G is (n, r, k)-critical, since $\frac{n}{r} = \frac{q^2}{r} > 64k(\log q)^{4k^2}q$ by the choice of n and q. Though it is not explicitly stated as a lemma, the following is proven in Lemma 2.2 of [10].

Lemma 4.7 ([10] Lemma 2.2*)**.** *Let $(V, \mathcal{L}')$ be the (random) hypergraph obtained by picking each hyperedge of $(V, \mathcal{L})$ with probability $\frac{\log^2 q}{q}$. Then $(V, \mathcal{L}')$ is* good *with probability at least $1/2 - o(1)$.*

To complete the proof of the lemma it would be enough to find r hypergraphs $\mathcal{L}_1, \ldots, \mathcal{L}_r$ which are *good* and satisfy that the hyperedges of different hypergraphs intersect in at most one vertex. To see this, let G_i be the graph associated with hypergraph $\mathcal{L}_i$. Then, as mentioned above, all the graphs G_i are (n, r, k)-critical. Furthermore, they are edge-disjoint, since for every i the edges of G_i lie inside hyperedges of $\mathcal{L}_i$ by (iii), and hyperedges of $\mathcal{L}_i$ and $\mathcal{L}_j$ intersect in at most one vertex (since they correspond to lines in the affine plane).

To find the r hypergraphs $\mathcal{L}_1, \ldots, \mathcal{L}_r$ which are *good*, choose a c-edge-colouring of $(V, \mathcal{L})$ at random, where $c := \frac{q}{\log^2 q}$. Note that, since $k \geqslant 3$ and by choice of q, c satisfies $c > 4r$. Let $\mathcal{L}_i$ be the sub-hypergraph in colour i ($1 \leqslant i \leqslant c$). Clearly, no two hypergraphs $\mathcal{L}_i$ and $\mathcal{L}_j$ contain the same hyperedge. Moreover, since hyperedges are lines in the affine plane, no two hyperedges intersect in more than one vertex. The probability that a line $\ell \in \mathcal{L}$ is in $\mathcal{L}_i$

is $\frac{\log^2 q}{q}$. So $\mathcal{L}_i$ has the same distribution as the random hypergraph $(V, \mathcal{L}')$ in Lemma 4.7. Therefore, $\mathcal{L}_i$ is *good* with probability at least $1/4$, provided q is large enough. Hence, the expected number of *good* hypergraphs $\mathcal{L}_i$ is at least $c/4 > r$. So, there exists a c-colouring of $(V, \mathcal{L})$ such that at least r of the monochromatic hypergraphs are *good*. After relabelling, we have the desired hypergraphs, finishing the proof of Lemma 4.3. $\qquad\square$

An upper bound polynomial in both k and r

Here, we prove Lemma 4.4. Let $r \geqslant 2, k \geqslant 3$. For $n \leqslant 8k^6 r^3$ we need to construct r (n, r, k)-critical graphs G_i on n vertices which are edge-disjoint. We will define incidence structures $\mathcal{I}_i = (\mathcal{P}, \mathcal{L}_i)$ on the same set of points such that the families of lines $\mathcal{L}_i$ are disjoint for distinct i. Further, any three lines within one $\mathcal{L}_i$ do not form a triangle. We will then, analogously to Dudek and Rödl [11], enrich the lines in $\mathcal{L}_i$ randomly, and show that the resulting graphs are edge-disjoint and each of them are (n, r, k)-critical with positive probability.

Proof of Lemma 4.4. First, let us define the incidence structures $\mathcal{I}$. Let q be the smallest prime power such that $k^2 r \leqslant q$, and let $\mathbb{F}_q$ be the finite field of order q. The common vertex set of our graphs is $V := \mathbb{F}_q^3$, i.e. $n = |V| \leqslant 8k^6 r^3$. For every $\lambda \in \mathbb{F}_q \setminus \{0\}$, we will define an incidence structure $\mathcal{I}_\lambda = (V, \mathcal{L}_\lambda)$ where $\mathcal{L}_\lambda$ is a family of lines in $\mathbb{F}_q^3$. For $\lambda \in \mathbb{F}_q \setminus \{0\}$ set

$$M_\lambda := \left\{ (1, \lambda\alpha, \lambda\alpha^2) : \alpha \in \mathbb{F}_q \setminus \{0\} \right\}.$$

We call M_λ the λ-moment curve. In [26], Wenger used the usual moment curve M_1 to construct dense C_6-free graphs. Note that for non-zero $\lambda_1 \neq \lambda_2$ the two curves M_{λ_1} and M_{λ_2} do not intersect. An important and crucial property is that, for any $\lambda \neq 0$, any three vectors from M_λ are linearly independent, that is for distinct $\alpha_1, \alpha_2, \alpha_3$,

$$\det \begin{pmatrix} 1 & \lambda\alpha_1 & \lambda\alpha_1^2 \\ 1 & \lambda\alpha_2 & \lambda\alpha_2^2 \\ 1 & \lambda\alpha_3 & \lambda\alpha_3^2 \end{pmatrix} = \lambda^2 (\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2)(\alpha_2 - \alpha_1) \neq 0.$$

In general, a line in $\mathbb{F}_q^3$ is a set of the form $\ell_{\mathbf{s}, \mathbf{v}} = \{\beta \mathbf{s} + \mathbf{v} : \beta \in \mathbb{F}_q\}$, where $\mathbf{s} \in \mathbb{F}_q^3 \setminus \{0\}$ is called the *slope*. We define

$$\mathcal{L}_\lambda := \{\ell_{\mathbf{s}, \mathbf{v}} : \mathbf{s} \in M_\lambda, \mathbf{v} \in \mathbb{F}_q^3\};$$

that is, in the incidence structure $\mathcal{I}_\lambda = (\mathbb{F}_q^3, \mathcal{L}_\lambda)$ we only allow lines with slope vectors from the λ-moment curve. Clearly, $|\mathcal{L}_\lambda| = |M_\lambda| \frac{q^3}{q} = q^2(q-1)$ since each line contains q points. We establish the following properties about each structure $\mathcal{I}_\lambda$, $\lambda \neq 0$.

(1) Every point $v \in V$ is contained in $q - 1$ lines from $\mathcal{L}_\lambda$ and every line $\ell \in \mathcal{L}_\lambda$ contains q points.

(2) Any two points lie in at most one line.

(3) No three lines in $\mathcal{L}_\lambda$ intersect pairwise in three distinct points (i.e. form a triangle).

Further, we have for $\lambda_1 \neq \lambda_2$,

(4) $\mathcal{L}_{\lambda_1} \cap \mathcal{L}_{\lambda_2} = \emptyset$.

For (1), note that every slope vector in M_λ gives rise to exactly one line through a given point $v \in V$. The second part of (1) follows from the definition of a line. Property (2) holds because lines are affine subspaces of dimension 1 in the vector space $\mathbb{F}_q^3$. For (3), suppose three lines in $\mathcal{L}_\lambda$ intersect pairwise in three distinct points. Then their three slope vectors would be linearly dependent, a contradiction to the linear independence of any three vectors in $\mathcal{L}_\lambda$ we established above. Property (4) simply follows from $M_{\lambda_1} \cap M_{\lambda_2} = \emptyset$ for $\lambda_1 \neq \lambda_2$.

Now, we are ready to define our graphs $G_1, \ldots, G_{q-1}$. Let $\lambda \in \mathbb{F}_q \setminus \{0\}$. We partition every line $\ell \in \mathcal{L}_\lambda$ randomly into k sets $L_1^{(\ell)}, \ldots, L_k^{(\ell)}$ each of cardinality $l_1 := \lfloor \frac{q}{k} \rfloor$ or $l_2 := \lceil \frac{q}{k} \rceil$. Note that $l_1, l_2 \geqslant rk$. To be precise, between all partitions of a line $\ell = \dot\bigcup_{j=1}^k L_j^{(\ell)}$ where $\left| L_1^{(\ell)} \right| = \cdots = \left| L_{k'}^{(\ell)} \right| = l_1$ and $\left| L_{k'+1}^{(\ell)} \right| = \cdots = \left| L_k^{(\ell)} \right| = l_2$ we choose one uniformly at random, choices for distinct lines in $\mathcal{L}_\lambda$ being independent. The graph G_λ on the vertex set $V = \mathbb{F}_q^3$ is defined as follows. For every $\ell \in \mathcal{L}_\lambda$ and any $i \neq j$, we include the edges of a complete bipartite graph between the vertex sets $L_i^{(\ell)}$ and $L_j^{(\ell)}$ on ℓ. That is, the graph G_λ consists of a collection of Turán graphs on q vertices with k parts. Each Turán part "lives" along one of the lines $\ell \in \mathcal{L}_\lambda$. By Property (2), these parts are edge-disjoint. Further, by Property (3), G_λ is K_{k+1}-free. Also, for distinct $\lambda \in \mathbb{F}_q^3$, by Property (4), the graphs G_λ are edge disjoint. To finish the proof, we show that for any fixed $\lambda \in \mathbb{F}_q \setminus \{0\}$ the graph G_λ is (n, r, k)-critical with positive probability. As the choices of the G_λ are done independently, there is a choice of $G_1, \ldots, G_{q-1}$ with the desired properties.

The calculations are similar to those in [11]. For a subset $W \subseteq V(G)$, let $\mathcal{A}(W)$ denote the event that $G_\lambda[W]$ contains no K_k. Let $U \subseteq V(G)$ be a subset of size $|U| = \lfloor \frac{n}{r} \rfloor$. Then, since by Property (3) any K_k can only appear within a line $\ell \in \mathcal{L}_\lambda$,

$$\mathcal{A}(U) \subseteq \bigcap_{\ell \in \mathcal{L}_\lambda} \mathcal{A}(U \cap \ell),$$

and therefore, since all the events $\mathcal{A}(U \cap \ell)$ are independent,

$$\mathbb{P}(\mathcal{A}(U)) \leqslant \prod_{\ell \in \mathcal{L}_\lambda} \mathbb{P}(\mathcal{A}(U \cap \ell)).$$

For a line $\ell \in \mathcal{L}_\lambda$, set $u_\ell := |U \cap \ell|$, and let $\ell = \bigcup_{j=1}^k L_j^{(\ell)}$ be the partition we chose at random. Then the event $\mathcal{A}(U \cap \ell)$ is equivalent to the existence of a $j \in [k]$ such that $U \cap L_j^{(\ell)} = \emptyset$. But, for fixed $j \in [k]$,

$$\mathbb{P}\left(U \cap L_j^{(\ell)} = \emptyset \right) = \frac{\binom{q-u_\ell}{|L_j^{(\ell)}|}}{\binom{q}{|L_j^{(\ell)}|}} \leqslant \left(1 - \frac{u_\ell}{q} \right)^{\left| L_j^{(\ell)} \right|} \leqslant \exp\left(-\frac{l_1 u_\ell}{q} \right).$$

Therefore,

$$\mathbb{P}(\mathcal{A}(U)) \leqslant \prod_{\ell \in \mathcal{L}_\lambda} \mathbb{P}\left(\exists j \in [k] : U \cap L_j^{(\ell)} = \emptyset\right)$$

$$\leqslant k^{|\mathcal{L}_\lambda|} \exp\left(-\sum_{\ell \in \mathcal{L}_\lambda} \frac{l_1 u_\ell}{q}\right)$$

$$= k^{|\mathcal{L}_\lambda|} \exp\left(-\frac{q-1}{q} l_1 |U|\right),$$

since every point in U belongs to exactly $q-1$ lines (Property (1)), and therefore $\sum_{\ell \in \mathcal{L}_\lambda} u_\ell = \sum_{\ell \in \mathcal{L}_\lambda} |U \cap \ell| = (q-1)|U|$. We obtain,

$$\mathbb{P}\left(\exists U \in \binom{V}{\lfloor \frac{n}{r}\rfloor} : \mathcal{A}(U)\right) \leqslant \binom{n}{\lfloor \frac{n}{r}\rfloor} k^{|\mathcal{L}_\lambda|} \exp\left(-\frac{q-1}{q} l_1 \left\lfloor \frac{n}{r}\right\rfloor\right)$$

$$\leqslant (re)^{n/r} k^{q^2(q-1)} \exp\left(-\frac{q-1}{q}(rk)\left\lfloor\frac{n}{r}\right\rfloor\right)$$

$$\leqslant \exp\left[q^3\left(\frac{\ln r}{r} + \frac{1}{r} + \ln k - \frac{3}{4}k\right)\right]$$

$$< 1$$

for $k \geqslant 3$ and $r \geqslant 3$. Therefore, there exists an instance of G_λ such that every subset U of size at least $\lfloor \frac{n}{r}\rfloor$ contains a K_k in G_λ. □

5 Concluding remarks

We have seen, as a consequence of Proposition 3.2 and Theorem 1.5, that $s_r(K_k) \geqslant s_{r-1}(K_k)$. However, it is not that clear that $s_r(K_k)$ is also increasing in k. We usually expect that graphs which are Ramsey for K_k should be "larger" than those which are Ramsey only for K_{k-1}. It would be quite unintuitive if the following conjecture was not true.

Conjecture 5.1. *For all $r \geqslant 3$, $k \geqslant 3$ we have that $s_r(K_k) \geqslant s_r(K_{k-1})$.*

We also saw that the Erdős-Rogers function is tightly connected to the study of $s_r(K_k)$. For our lower bounds in Section 3, we essentially showed that $P_r(k) = \Omega\left(r(f_{k,k+1}(r))^2\right)$, provided $g_k(n) = \frac{f_{k,k+1}(n)}{\sqrt{n}}$ is any decent polylogarithmic function (which we believe it is). On the other hand, we saw in Section 4 that the known constructions for K_{k+1}-free graphs with small k-independence number can be modified to constructions of r pairwise edge-disjoint such graphs on the same or just slightly larger vertex set. In fact, if a packing of essentially optimal (n, r, k)-graphs G, that is, those with parameters $n/r = \Theta(\alpha_k(G)) = \Theta(f_{k,k+1}(n))$, was possible then we would get an upper bound that matches our lower bounds. Indeed, then $\sqrt{n} = \Theta(r \cdot g_k(n)) = \Theta(r \cdot g_k(r)) = \Theta(\sqrt{r} \cdot f_{k,k+1}(r))$, so by Lemma 4.1 we would have

$$P_r(k) \leqslant n = \Theta\left(r(f_{k,k+1}(r))^2\right).$$

We strongly believe the following is true.

Conjecture 5.2. *For every fixed $k \geqslant 3$,*

$$s_r(K_k) = \Theta\left(r \cdot (f_{k-1,k}(r))^2\right).$$

Therefore, we believe that tightening the known bounds on $f_{k-1,k}(n)$ will directly contribute to tightening the bounds on $s_r(K_k)$. The currently best known bounds [10] on the Erdős-Rogers function are

$$\Omega\left(\sqrt{\frac{n \ln n}{\ln \ln n}}\right) = f_{k,k+1}(n) = O\left((\ln n)^{4k^2}\sqrt{n}\right),$$

so it is not yet clear how strongly the logarithmic factor depends on k. We wonder whether the upper bound can be strengthened in the following way.

Question 5.3. *Does there exist a universal constant C (independent of k) such that $f_{k,k+1}(n) = O\left((\ln n)^C \sqrt{n}\right)$? And does the construction of such a K_{k+1}-free graph on n vertices with k-independence number less than $O\left((\ln n)^C \sqrt{n}\right)$ generalise to a packing of such graphs?*

A positive answer to both questions would imply that there is a universal constant $C > 0$ such that $s_r(K_k) = O(r^2(\ln r)^C)$.

In the special case of K_3, in the proof of Lemma 4.2 we iteratively applied the Local Lemma to find edge-disjoint triangle-free subgraphs $G_i \subseteq K_n$ with independence number $O(\sqrt{n} \ln n)$ and this implied our upper bound in Theorem 1.1. This approach was an adaptation of the classical application of the Local Lemma by Spencer [24] to lower bound off-diagonal Ramsey numbers and obtain $R(3, k) \geqslant c \left(k/\ln k\right)^2$. Subsequently Kim [17] proved the existence of a triangle-free graph G on n vertices with independence number $O\left(\sqrt{n \ln n}\right)$, hence establishing that correct order of magnitude of $R(3, k)$ is $k^2/\ln k$. Earlier Bollobás and Erdős suggested an alternative approach to the problem of finding better lower bounds on $R(3, k)$: the triangle-free process. In 2009, Bohman [3] managed to reprove Kim's theorem using the triangle-free process. Very recently, Fiz Pontiveros, Griffiths and Morris [15], and independently Bohman and Keevash [4], improved the constant factor in the analysis and showed that $R(3, k) \geqslant (1/4 - o(1))k^2/\ln k$. We are optimistic that one can apply the triangle-free process iteratively, with some modifications, and thus find not only one, but a packing of triangle-free graphs $G_1, \ldots, G_r$ on n vertices, all having independence number $O(\sqrt{n \ln n})$. Thus, we conjecture that our lower bound on $s_r(K_3)$ is tight.

Conjecture 5.4. $s_r(K_3) = \Theta\left(r^2 \ln r\right)$.

References

[1] M. Ajtai, J. Komlós, and E. Szemerédi. A note on Ramsey numbers. *J. Combin. Theory Ser. A*, **29**:354–360, 1980.

[2] N. Alon and J. Spencer. *The probabilistic method.* Wiley-Interscience, 2008.

[3] T. Bohman. The triangle-free process. *Adv. Math.*, **221**:1653–1677, 2009.

[4] T. Bohman and P. Keevash. Dynamic concentration of the triangle-free process. *preprint*, 2013. available at `http://arxiv.org/pdf/1302.5963v1`.

[5] B. Bollobás and H. Hind. Graphs without large triangle free subgraphs. *Discrete Math.*, **87**:119–131, 1991.

[6] S. Burr, P. Erdős, and L. Lovász. On graphs of Ramsey type. *Ars Combin.*, **1**:167–190, 1976.

[7] S. A. Burr, J. Nešetřil, and V. Rödl. On the use of senders in generalized Ramsey theory for graphs. *Discrete Math.*, **54**:1–13, 1985.

[8] D. Conlon. A new upper bound for diagonal Ramsey numbers. *Ann. of Math.*, **170**:941–960, 2009.

[9] A. Dudek and D. Mubayi. On generalized Ramsey numbers for 3-uniform hypergraphs. *J. Graph Theory*, **76**:217–223, 2013.

[10] A. Dudek, T. Retter, and V. Rödl. On generalized Ramsey numbers of Erdős and Rogers. *Journal of Combinatorial Theory, Series B*, 109:213–227, 2014.

[11] A. Dudek and V. Rödl. On K_s-free subgraphs in K_{s+k}-free graphs and vertex Folkman numbers. *Combinatorica*, **31**:39–53, 2011.

[12] P. Erdős and C. Rogers. The construction of certain graphs. *Canad. J. Math*, **14**:702–707, 1962.

[13] P. Erdős. Some remarks on the theory of graphs. *Bull. Amer. Math. Soc.*, **53**:292–294, 1947.

[14] P. Erdős and G. Szekeres. A combinatorial problem in geometry. *Compositio Math.*, **2**:463–470, 1935.

[15] G. Fiz Pontiveros, S. Griffiths, and R. Morris. The triangle-free process and $R(3, k)$. *preprint*, 2013. available at `http://arxiv.org/pdf/1302.6279v1`.

[16] J. Fox and K. Lin. The minimum degree of Ramsey-minimal graphs. *J. Graph Theory*, **54**:167–177, 2006.

[17] J. H. Kim. The Ramsey number $R(3, t)$ has order of magnitude $t^2 / \log t$. *Random Structures Algorithms*, **7**:173–207, 1995.

[18] M. Krivelevich. K_s-free graphs without large K_r-free subgraphs. *Combin. Probab. Comput.*, **3**:349–354, 1994.

[19] F. P. Ramsey. On a problem of formal logic. *Proc. London Math. Soc.*, **30**:264–286, 1930.

[20] V. Rödl and M. Siggers. On Ramsey minimal graphs. *SIAM J. Discrete Math.*, **22**:467–488, 2008.

[21] J. B. Shearer. A note on the independence number of triangle-free graphs. *Discrete Math.*, **46**:83–87, 1983.

[22] J. B. Shearer. On the independence number of sparse graphs. *Random Structures Algorithms*, **7**:269–271, 1995.

[23] J. Spencer. Ramsey's theorem – a new lower bound. *J. Combin. Theory Ser. A*, **18**:108–115, 1975.

[24] J. Spencer. Asymptotic lower bounds for Ramsey functions. *Discrete Math.*, **20**:69–76, 1977.

[25] H. Wan. Upper bounds for Ramsey numbers $R(3, 3, ..., 3)$ and Schur numbers. *J. Graph Theory*, **26**:119–122, 1997.

[26] R. Wenger. Extremal graphs with no C_4's, C_6's, or C_{10}'s. *J. Combin. Theory Ser. B*, **52**:113–116, 1991.

[27] X. Xiaodong, X. Zheng, G. Exoo, and S. Radziszowski. Constructive lower bounds on classical multicolor Ramsey numbers. *Electron. J. Combin.*, **11**:R35, 2004.