There exist graphs with super-exponential Ramsey multiplicity constant

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Abstract

The Ramsey multiplicity $M(G;n)$ of a graph G is the minimum number of monochromatic copies of G over all 2-colorings of the edges of the complete graph K_n. For a graph G with a automorphisms, v vertices, and E edges, it is natural to define the *Ramsey multiplicity constant* $C(G)$ to be $\lim_{n\to\infty} \frac{M(G;n)a}{v!\binom{n}{v}}$, which is the limit of the fraction of the total number of copies of G which must be monochromatic in a 2-coloring of the edges of K_n. In 1980, Burr and Rosta showed that $0 < C(G) \leq 2^{1-E}$ for all graphs G, and conjectured that this upper bound is tight. Counterexamples of Burr and Rosta's conjecture were first found by Sidorenko and Thomason independently. Later, Clark proved that there are graphs G with E edges and $2^{E-1}C(G)$ arbitrarily small. We prove that for each positive integer E there is a graph G with E edges and $C(G) \leq E^{-E/2+o(E)}$.

1 Introduction

The *Ramsey number* of a graph G, denoted $R(G)$, is the minimum positive integer n such that every 2-coloring of the edges of K_n contains a monochromatic copy of G. A famous 1930 theorem of Ramsey guarantees the existence of $R(G)$. Let $M(K_3,n)$ denote the minimum number of monochromatic triangles over all 2-colorings of the edges of K_n. In 1959, Goodman [10] found an exact formula for $M(K_3,n)$:

$$M(K_3,n) = \begin{cases} \frac{n(n-2)(n-4)}{24} & \text{if } n \text{ is even} \\ \frac{n(n-1)(n-5)}{24} & \text{if } n \equiv 1 \pmod 4 \\ \frac{(n+1)(n-3)(n-4)}{24} & \text{if } n \equiv 1 \pmod 4 \end{cases} \tag{1}$$

Goodman mentions in the last section of this early paper that, "It seems obvious that the methods used here should generalize to answer questions about the minimum number of full and empty quadrilaterals, and figures of a higher number of sides, but up to the present, I have not been able to carry through the computations successfully." The *Ramsey multiplicity* $M(G;n)$ of a graph G is the minimum number of monochromatic copies of G over all 2-colorings of the edges of K_n. In 1962, Erdős [5], using the probabilistic method, gave an upper bound on the Ramsey multiplicity of a complete graph:

$$M(K_v;n) \leq \binom{n}{v} 2^{1-\binom{v}{2}}. \tag{2}$$

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Erdős conjectured that this value is asymptotically tight: $\lim_{n\to\infty} \frac{M(K_v;n)2^{\binom{v}{2}-1}}{\binom{n}{v}} = 1$. Restated, Erdős's conjecture says that a uniform random coloring has asymptotically the same number of monochromatic copies of the complete graph K_v as a coloring with the minimum number of monochromatic copies of K_v. Goodman's theorem implies Erdős's conjecture is true for $v = 3$.

Burr and Rosta [2] in 1980 wrote a survey on Ramsey multiplicity. Let $a = a(G)$ be the number of automorphisms of G, $v = v(G)$ be the number of vertices of G, and $E = e(G)$ be the number of edges of G. By a simple counting argument we can show that the total number of copies of G in K_n is $\frac{v!}{a}\binom{n}{v}$. Let $C(G;n)$ be $\frac{aM(G;n)}{v!\binom{n}{v}}$, the fraction of the total number of copies of G which must be monochromatic in a 2-coloring of the edges of K_n. Burr and Rosta showed that in a random 2-coloring of the edges of K_n with each coloring equally likely, the expected number of monochromatic copies of G is $\frac{2^{1-E}v!}{a}\binom{n}{v}$. It follows that for all graphs G, $C(G;n) \leq 2^{1-E}$. This upper bound on the Ramsey multiplicity we call the *random bound*. They also showed that $C(G;n)$ is a monotonic increasing function of n. It is therefore natural to define the *Ramsey multiplicity constant* $C(G)$ by

$$C(G) := \lim_{n\to\infty} C(G;n).$$

In words, the Ramsey multiplicity constant is the asymptotic fraction (as $n \to \infty$) of copies of G that must be monochromatic in 2-colorings of the edges of K_n. It follows that $0 < C(G) \leq 2^{1-E}$.

Erdős's conjecture is equivalent to $C(K_v) = 2^{1-\binom{v}{2}}$. After it was shown for several graphs that $C(G) = 2^{1-e(G)}$, Burr and Rosta [2] made the stronger conjecture that $C(G) = 2^{1-e(G)}$ holds for all graphs G. It is known that $C(G) = 2^{1-e(G)}$ is true for forests, cycles, and wheels with an even number of spokes [2, 3, 4, 10, 12, 13, 14, 15].

In 1989, Thomason [17] disproved Erdős conjecture by showing for $G = K_t$, that $C(G) < (.976)2^{1-e(G)}$ for $t = 4$, $C(G) < (.906)2^{1-e(G)}$ for $t = 5$, and $C(G) < (.936)2^{1-e(G)}$ for $t \geq 6$. These upper bounds were improved by better constant factors and simplified in several papers [7, 8, 9, 18].

Jagger, Stovicek, and Thomason [13] proved that if G contains a K_4 subgraph, then $C(G) < 2^{1-e(G)}$ holds. However, Sidorenko [16] has conjectured that $C(G) = 2^{1-e(G)}$ is true for bipartite G.

Burr and Rosta's conjecture was disproved in a strong sense by Clark [3], who showed that if G is a connected graph with E vertices and E edges that contains a triangle, then $C(G) \leq (5\sqrt{\frac{\log_2(E)}{E}})2^{1-E}$ holds.

The following theorem is proved in Section 2.

Theorem 1.1 *For each positive integer E, there is a graph G with E edges and*

$$C(G) \leq E^{-E/2+o(E)}.$$

where the $o(E)$ term goes to 0 as $E \to \infty$.

We conjecture the following weak form of Burr and Rosta's conjecture.

Conjecture 1.2 *For all graphs G, we have*

$$C(G) = 2^{-e(G)^{1+o(1)}} \tag{3}$$

where the $o(1)$ term goes to 0 as $e(G) \to \infty$.

In Section 4 we show how a recent result of Alon, Krivelevich, and Sudakov [1] on a conjecture of Erdős concerning Ramsey numbers implies that $C(G) \geq 2^{-e(G)^{3/2+o(1)}}$. In Section 5 we discuss the generalization of Ramsey multiplicity of graphs for edge colorings using r colors.

2 Warm Up

We let $\chi(G)$ denote the chromatic number of G. The following lemma is the main ingredient in the proof of Theorem 1.1.

Lemma 2.1 *For all connected graphs G with at least one edge, we have*

$$C(G) \leq (\chi(G) - 1)^{1-v(G)}.$$

Proof. Let $k = \chi(G) - 1$, $v = v(G)$, and $a = a(G)$. Consider the 2-coloring of the edges of K_{kn} for which the blue edges form k disjoint cliques on n vertices, and the other edges are colored red. Since the red edges form a k-partite graph, and the graph G has chromatic number $k+1$, then there are no monochromatic red copies of G in this 2-coloring. In each of the k blue copies of K_n, there are exactly $\frac{v!}{a}\binom{n}{v}$ monochromatic copies of G. Hence,

$$C(G) \leq \lim_{n\to\infty} k\frac{v!}{a}\binom{n}{v} / \left(\frac{v!}{a}\binom{kn}{v}\right) = k^{1-v}.$$

□

The *Lollipop graph* $L(m,t)$ is the graph formed by taking a complete graph K_m and connecting a pendant path of length t to one of the m vertices of the complete graph.

To finish the proof of Theorem 1.1, we fix a positive integer E and let $G = L(m,t)$ where $m = \lfloor\sqrt{\frac{2E}{\log E}}\rfloor$ and $t = E - \binom{m}{2}$. We have $v(G) = m + t = (1 + o(1))E$ and $\chi(G) = m = E^{1/2+o(1)}$. By Lemma 2.1,

$$C(G) \leq (\chi(G) - 1)^{1-v(G)} = E^{-E/2+o(E)}.$$

The upper bound on the Ramsey multiplicity constant given by Lemma 2.1 improves considerably on the random bound for connected sparse graphs with large enough chromatic number. It is self-evident that the Lollipop graphs with $t > \frac{m^2}{\log m}$ and $m \geq 4$ fit this description.

In [13], Jagger, Stovicek, and Thomason call a graph G *common* if Burr and Rosta's conjecture holds for G, that is, if $C(G) = 2^{1-e(G)}$. If a graph isn't common, then it is called *uncommon*. Jagger, Stovicek, and Thomason [13] conjectured that every graph G with $\chi(G) \geq 4$ is uncommon. It follows from Lemma 2.1 that their conjecture is true when G is connected and sufficiently sparse. To be precise, if G is connected and $e(G) < 1 + (v(G) - 1)\log_2(\chi(G) - 1)$, then G is uncommon.

We show further that for each $\chi \geq 3$, there exists a family of graphs with chromatic number χ and Ramsey multiplicity constant at least an exponential factor smaller than given by the random bound.

Theorem 2.2 *For each integer $\chi \geq 3$, there is a constant c_χ with $c_\chi < \frac{1}{2}$ such that there are graphs G with chromatic number χ, E edges, and $C(G) = O(c_\chi^E)$. In particular, we may take $c_3 = .4996155$ and $c_\chi = \frac{1}{\chi-1}$ for $\chi \geq 4$.*

Theorem 2.2 follows from Lemma 2.1 in the case $\chi \geq 4$ by letting G be the Lollipop graph $L(\chi, e - \binom{\chi}{2})$. The case $\chi = 3$ is particularly interesting because there are several families of graphs with chromatic number 3 that are known to be common, such as odd cycles [15] and wheels with an even number of spokes [13]. The proof of Theorem 2.2 when $\chi = 3$ uses Theorem 3.1, a generalization of Lemma 2.1.

3 A general upper bound

We say that G is *formed by adjoining the forest F to the graph H* if the edges of G are those of both F and H, and each connected component of F has exactly one vertex in common with H. Jagger et al. [13] proved that the graph formed by adjoining a sufficiently large tree to a non-bipartite graph is an uncommon graph. We will go further and give a general upper bound on the Ramsey multiplicity constant of graphs formed by adjoining a forest to a connected graph.

Let $f(G, k)$ denote the minimum number of edges in G with vertices the same color over all k-colorings of the vertices of G. We define the *random graph $\mathcal{G}(n, p)$* to be a probability space over the set of graphs on n vertices determined by each edge having probability p of existing, mutually independent of each other.

Theorem 3.1 gives a general bound on the Ramsey multiplicity constant of a graph based on basic properties of the graph. In the proof of Theorem 3.1, we count labeled copies of G instead of unlabeled copies for G. This does not affect the value of $C(G)$.

Theorem 3.1 *Let G be a graph formed by adjoining a forest F to a connected graph H. If $2 \leq k < \chi(G)$ and $0 \leq p \leq 1$, then*

$$C(G) \leq p^E k^{1-v} + (1-p)^f (1 - \frac{p}{k})^{v-v'} \tag{4}$$

where $E = e(G)$, $v = v(G)$, $v' = v(H)$, and $f = f(H, k)$.

Proof. We first find the expected number of labeled copies of G in $\mathcal{G}(n, p)$. There are $\binom{n}{v}$ choices of v vertices of our n vertices, $v!$ different orderings of these v vertices, and a probability p^E that each edge of each ordering of G on these vertices is present in $\mathcal{G}(n, p)$. This gives an expectation of $v!\binom{n}{v}p^E$ for the number of labeled copies of G in $\mathcal{G}(n, p)$.

Let $k\mathcal{G}(n, p)$ consist of k disjoint copies of $\mathcal{G}(n, p)$, in other words, of k disjoint n vertex random graphs. Let G_2 denote the complement of $k\mathcal{G}(n, p)$ in K_{kn}, whose graphs consists of kn vertices in k parts of size n such that the probability of an edge existing within a part is $1 - p$ and between parts is 1.

Because G is connected, then every copy of G in $k\mathcal{G}(n, p)$ is contained within one of the k copies of $\mathcal{G}(n, p)$. By linearity of expectation, the expected number of labelled copies of G in $k\mathcal{G}(n, p)$ is $kv!\binom{n}{v}p^E$.

We now consider the expected number of copies of G in G_2. The vertices of H in a copy of G must be assigned to the k parts in some manner and by definition this puts at least $f(H,k)$ edges of H within the various parts. This means that the probability of a copy of H being in G_2 is at most $(1-p)^f$. The edges of $G-H$ each lead from H to a new vertex. These edges have probability $1-p$ if the new vertex is in the same part as the one it is attached to, and probability 1 otherwise. This produces a probability asymptotic to $\frac{(1-p)n+(k-1)n}{kn} \sim 1-\frac{p}{k}$ for each of the $v(G)-v(H)$ edges in F to exist in G_2. The probability that all such edges as well as those of H are in G_2 is therefore at most $(1-p)^{f(H)}(1-\frac{p}{k})^{v(G)-v(H)}$. Therefore, we have

$$C(G) \le \lim_{n\to\infty} \frac{kv!\binom{n}{v}p^E}{v!\binom{kn}{v}} + (1-p)^f(1-\frac{p}{k})^{v-v'} = p^E k^{1-v} + (1-p)^f(1-\frac{p}{k})^{v-v'}.$$

□

We note that Lemma 2.1 is the case $p=1$ and $k=\chi(G)-1$ in Theorem 3.1.

The bound given in Theorem 3.1 on the Ramsey multiplicity constant is strongly dependent on $\chi(G)$. This fits nicely with Sidorenko's conjecture that $C(G)=2^{1-e(G)}$ for bipartite graphs G.

We will need the following lemma to prove Theorem 2.2 in the case $\chi=3$.

Lemma 3.2 *For all graphs G with v vertices and chromatic number k, we have*

$$f(G,k-1) \le \lfloor \frac{v}{k} \rfloor \lfloor \frac{v+1}{k} \rfloor.$$

This upper bound is attained when $G=K_{t_1,t_2,\ldots,t_k}$, where $t_i=\lfloor\frac{v+i-1}{k}\rfloor$.

Proof. To prove this upper bound combine the two smallest color classes in a proper k-coloring of the vertices of G. We now prove that this upper bound is achieved for the complete k-partite graph $K_{t_1,t_2,\ldots,t_k}$. Consider the partition of the vertex set of G: $V(G)=V_1\cup V_2\cup\cdots\cup V_k$ with V_i denoting the vertex class with t_i independent vertices. Every copy of K_k in $K_{t_1,t_2,\ldots,t_k}$ has exactly one vertex in each of the k vertex classes, so that the number of copies of K_k in $K_{t_1,t_2,\ldots,t_k}$ is $\prod_{i=1}^k t_i$. In every $(k-1)$-coloring of the vertices of a copy of K_k, there exists at least one edge such that its vertices are the same color. For $i\ne j$ and every edge (v,w) of $K_{t_1,t_2,\ldots,t_k}$ with $v\in V_i$ and $w\in V_j$, the number of copies of K_k in G that contain v and w is $\frac{1}{t_it_j}\prod_{i=1}^k t_i \le \frac{1}{t_1t_2}\prod_{i=1}^k t_i$. Therefore, $f(G,k-1)\ge \left(\prod_{i=1}^k t_i\right)\left(\frac{1}{t_1t_2}\prod_{i=1}^k t_i\right)^{-1} = t_1t_2$. □

We now prove Theorem 2.2 with $\chi=3$. The proof relies on Theorem 3.1, using $p=.997503$ and $k=\chi-1=2$. Let $L(n,t)$ be the graph formed by adjoining a path of length t to a vertex of a complete 3-partite graph $H=K_{n,n,n}$. It follows that $v(L(n,t))=3n+t$, $e(L(n,t))=3n^2+t$, $\chi(L(n,t))=3$, and $v(L(n,t))-v(H)=t$. From Lemma 3.2, we have $f(K_{n,n,n},2)=n^2$. Let $p=.997503$ and $t=\lfloor\frac{3pn^2}{1-p}\rfloor$. We find $t=(p+o(1))E$, $v(L(n,t))=(p+o(1))E$, and $f(L(n,t),2)=(\frac{1-p}{3}+o(1))E$. Substituting into Theorem 3.1, yields

$$C(L(n,t)) \le p^E 2^{(-p+o(1))E} + (1-p)^{(\frac{1-p}{3}+o(1))E}(1-\frac{p}{2})^{(p+o(1))E} = O(.4996155^E).$$

This settles Theorem 2.2 in the case $\chi=3$.

4 Ramsey numbers and Ramsey multiplicity

In this section we consider how small the Ramsey multiplicity constant $C(G)$ can be as a function of the number of edges $e(G)$. In doing so, we relate a couple of longstanding conjectures of Erdős on Ramsey numbers to the Ramsey multiplicity of graphs.

Conjecture 4.1 (Erdős 1983) *There exists an absolute constant c such that the Ramsey number $R(G)$ obeys*

$$R(G) < 2^{cE^{\frac{1}{2}}}$$

for all graphs G without isolated vertices and E edges.

Erdős's conjecture [6] was recently positively settled for bipartite graphs [1]. In the same paper, Alon, Krivelevich, and Sudakov [1] proved a slightly weaker result than Erdős's conjecture in general. They proved that there exists an absolute constant $c > 0$ such that for all graphs G without isolated vertices, we have

$$R(G) < 2^{ce(G)^{\frac{1}{2}} \log e(G)}.$$

We use this recent upper bound on Ramsey numbers to give a lower bound on the Ramsey multiplicity constant.

Lemma 4.2 *There exists an absolute constant $C > 0$ such that for all connected graphs G,*

$$C(G) \geq 2^{-CE^{\frac{3}{2}} \log E} \tag{5}$$

holds, where $E = e(G)$.

Proof. Since $M(G; R(G)) \geq 1$ and $C(G; n)$ is a monotonic increasing function of n, then

$$C(G) \geq C(G; R(G)) \geq \frac{a}{v!} \binom{R(G)}{v}^{-1} \geq R(G)^{-v} \geq 2^{-cvE^{\frac{1}{2}} \log E} \geq 2^{-2cE^{\frac{3}{2}} \log E}.$$

Taking $C = 2c$, we arrive at the desired result. □

If Conjecture 1.2 is true, then the lower bound in Lemma 4.2 is far from correct. The lower bound on the Ramsey multiplicity in Lemma 4.2 is terrible for dense graphs, as $v(G)$ is usually on the order of $\sqrt{e(G)}$ and not $e(G)$. The following result shows that dense graphs have Ramsey multiplicity constant that is exponential in the number of edges of the graph, and so they are not good examples of graphs with small Ramsey multiplicity constant in terms of the number of edges of the graph.

Lemma 4.3 *If $e(G) = d\binom{v(G)}{2}$, then $C(G) \geq 2^{\frac{-4e(G)}{d}}$.*

Proof. Using the well known [11] upper bound $R(K_v) \leq 4^{v-1}$ and following the same proof as Lemma 4.2, we get

$$C(G) \geq R(G)^{-v} \geq R(K_v)^{-v} \geq 2^{-4\binom{v}{2}} \geq 2^{\frac{-4e(G)}{d}}.$$

□

5 Ramsey multiplicity with multiple colors

In this section we consider the natural generalization to edge colorings using r colors, where $r \geq 2$. The *Ramsey number* $R(G;r)$ is the smallest integer n for which every r-coloring of the complete graph K_n contains a monochromatic copy of G. The *Ramsey multiplicity* $M_r(G;n)$ is the minimum number of monochromatic copies of G over all r-colorings of the edges of K_n. Many of the lemmas mentioned above have straightforward generalizations from 2 to r colors. For example, we have $M_r(G;n) \leq \frac{v!}{r^{e-1}a}\binom{n}{v}$, since this upper bound is the expected number of monochromatic copies of G in a random r-coloring of the edges of K_n where each r-coloring is equally likely. Also, $C_r(G;n) = \frac{M_r(G;n)}{\frac{v!}{a}\binom{n}{v}}$ is a monotonic increasing function of n. Therefore the limit $C(G;r) = \lim_{n\to\infty} C_r(G;n)$ exists and $0 < C(G;r) \leq r^{1-e}$.

Jagger et al. [13] showed that for every non-bipartite graph G, there is positive integer r_0 for which $C(G;r) < r^{1-e}$ for all $r \geq r_0$. While the random bound is the reciprocal of a polynomial growth in r, the theorem below gives a reciprocal of an exponential growth in r upper bound on the Ramsey multiplicity constant.

Theorem 5.1 *If G is a connected graph with at least one edge, then*

$$C(G;r) \leq (\chi(G)-1)^{(r-1)(1-v)}$$

Proof. Partition the vertices of K_n into $(\chi(G)-1)^{r-1}$ vertex classes $V_1, \ldots, V_{(\chi(G)-1)^{r-1}}$ of as equal size as possible, so $|V_i - V_j| \leq 1$ for $1 \leq i \leq j \leq (\chi(G)-1)^{r-1}$. If $v, w \in V_i$, then color the edge (v,w) the color $r-1$. If $i \neq j$ and $v \in V_i$ and $w \in V_j$, color the edge (v,w) the color c, where c is the largest power of $(\chi(G)-1)$ that divides $i-j$. Each color class other than that of the color $r-1$ is the union of disjoint complete $(\chi(G)-1)$-partite graphs and therefore contains no monochromatic copies of G. The graph of color $r-1$ consists of $(\chi(G)-1)^{r-1}$ disjoint complete graphs each with at most $\lceil \frac{n}{(\chi(G)-1)^{r-1}} \rceil$ vertices. Therefore, there are

$$(1+o(1))(\chi(G)-1)^{(r-1)(1-v)}\frac{v!\binom{n}{v}}{a}$$

monochromatic copies of G. Letting $n \to \infty$, we have $C(G;r) \leq (\chi(G)-1)^{(r-1)(1-v)}$. □

For complete graphs, the following theorem demonstrates a strong dependence between the Ramsey multiplicity $C(K_k;r)$ and the Ramsey number $R(K_k;r)$ for k fixed and r a variable.

Theorem 5.2 *For all positive integers $k, r \geq 2$,*

$$\binom{R(K_k;r)}{k}^{-1} \leq C(K_k;r) \leq (R(K_k;r-1)-1)^{1-k}.$$

Proof. The lower bound on $C(K_k;r)$ follows from noting that $\frac{M_r(K_k;n)}{\binom{n}{k}}$ is a monotonic increasing function of n and $M_r(K_k;R(K_k;r)) \geq 1$. The upper bound on $C(K_k;r)$ comes from a specific coloring. Consider an $(r-1)$-coloring with colors $1, \ldots, r-1$ of the edges of $K_{R(K_k;r-1)-1}$ with vertices

$1, \ldots, R(K_k; r-1)-1$ without a monochromatic K_k, and let $c(i,j)$ be the color of the edge (i,j). Partition the vertices of $K_{(R(K_k;r-1)-1)n}$ into $R(K_k; r-1)-1$ classes of n vertices each: $V_1, \ldots, V_{R(K_k;r-1)-1}$. If $i \neq j$ and $v \in V_i$, $w \in V_j$, then assign the color $c(v,w) = c(i,j)$ to the edge (v,w). If $v, w \in V_i$ for some i, then assign the color $c(v,w) = r$ to the edge (v,w). So the only monochromatic K_k are the color r, and the graph formed by color class r consists of $R(K_k; r-1) - 1$ disjoint copies of K_n. Letting $n \to \infty$, it follows that $C(K_k; r) \leq (R(K_k; r-1) - 1)^{1-k}$. □

Conjecture 5.3 is another outstanding Erdős conjecture on Ramsey numbers.

Conjecture 5.3 (\$100 **Erdős Problem**) *We have* $\lim_{r\to\infty} R(K_3; r)^{1/r} = \infty$.

It follows from Theorem 5.2 that Conjecture 5.3 is equivalent to $\lim_{r\to\infty} C(K_3; r)^{1/r} = 0$. The following conjecture is a natural generalization of Conjecture 5.3.

Conjecture 5.4 *For all non-bipartite graphs G,*

$$\lim_{r\to\infty} C(G; r)^{-1/r} = \lim_{r\to\infty} R(G; r)^{1/r} = \infty.$$

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