MAT 307: Combinatorics

Lecture 20: Friends and politicians

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1 The friendship theorem

Theorem 1. *Suppose G is a (finite) graph where any two vertices share exactly one neighbor. Then there is a vertex adjacent to all other vertices.*

The interpretation of this theorem is as follows: if any two people have exactly one friend in common, then there is a person (the politician) who is everybody's friend. We actually prove a stronger statement, namely that the only graph with this structure consists of a collection of triangles that all share one vertex.

Surprisingly, the friendship theorem is false for infinite graphs. Let $G_0 = C_5$, and let G_{n+1} be obtained from G_n by adding a separate common neighbor to each pair of vertices that does not have one yet. Then $G = \bigcup_{n=0}^{\infty} G_n$ is a counterexample to the theorem.

The theorem for finite graphs sounds somewhat similar to the Erdős-Ko-Rado theorem. Interestingly, the proof requires some spectral analysis.

Proof. Assume for the sake of contradiction that any two vertices in G share exactly one vertex, but there is no vertex adjacent to all other vertices. Note that the first condition implies that there is no C_4 subgraph in G.

First, we claim that G is a regular graph. Suppose $(u, v) \notin E$ and $w_1, \ldots, w_k$ are the neighbors of u. We know that v and w_i share a neighbor z_i for each i. The vertices z_i must be distinct, otherwise we would get a C_4 (between u, w_i, w_j and $z_i = z_j$). Therefore, v also has at least k neighbors. By symmetry, we conclude that $deg(u) = deg(v)$ for any $(u, v) \notin E$. Assuming that w_1 is the only shared neighbor of u and v, any other vertex w is adjacent to at most one of u, v and hence $deg(w) = deg(u) = deg(v)$. Finally, w_1 is not adjacent to all these vertices, so $deg(w_1) = deg(u) = deg(v)$ as well.

Hence, all degrees are equal to k. The number of walks of length 2 from a fixed vertex x is k^2. Because every vertex $y \neq x$ has a unique path of length 2 from x, this way we count every vertex once except x itself, which is counted k times. To account for that, we subtract $k-1$ and the total number of vertices is $n = k^2 - k + 1$.

We consider the adjacency matrix A. Since any two vertices share exactly one neighbor, the matrix A^2 has k on the diagonal and 1 everywhere else. We can write

$$A^2 = J + (k-1)I.$$

From this expression, it's easy to see that A^2 has eigenvalues $n+k-1 = k^2$, and $k-1$ of multiplicity $n-1$. The eigenvalues of A^2 are squares of the eigenvalues of A, which are k (the degree of each vertex), and $\pm\sqrt{k-1}$. We know that the eigenvalues should sum up to 0. If $\sqrt{k-1}$ appears with multiplicity r and $-\sqrt{k-1}$ appears with multiplicity s. This yields

$$k + (r-s)\sqrt{k-1} = 0.$$

This implies $k^2 = (s-r)^2(k-1)$, i.e. $k-1$ divides k^2. This is possible only for $k = 1, 2$; otherwise, $k-1$ divides $k^2 - 1$ and hence cannot divide k^2. For $k = 1, 2$, we get two regular graphs: K_1 and K_3. These both satisfy the conditions of the theorem and also the conclusion.

Otherwise, we conclude that there must be a vertex x adjacent to all other vertices. Then it's easy to see that these vertices are matched up and form triangles with the vertex x. □

We finish with a related conjecture of Kotzig.

Conjecture. *For any fixed $\ell > 2$, there is no finite graph such that every pair of vertices is connected by precisely one path of length ℓ.*

For $\ell = 2$, we concluded that there is exactly one such graph - a collection of triangles joined by one vertex. This conjecture has been verifed for $3 \leq \ell \leq 33$, but a general proof remains elusive.

2 The variational definition of eigenvalues

We continue with an equivalent definition of eigenvalues.

Lemma 1. *The k-th largest eigenvalue of a matrix $A \in R^{n \times n}$ is equal to*

$$\lambda_k = \max_{dim(\mathcal{U})=k} \min_{x \in \mathcal{U}} \frac{x^T A x}{x^T x} = \min_{dim(\mathcal{U})=k-1} \max_{x \perp \mathcal{U}} \frac{x^T A x}{x^T x}.$$

Here, the maximum/minimum is over all subspaces $\mathcal{U}$ of a given dimension, and over all nonzero vectors x in the respective subspace.

Proof. We only prove the first equality - the second one is analogous. First of all, note that the quantity $x^T Ax/x^T x$ is invariant under replacing x by any nonzero multiple μx. Therefore, we can assume that x is a unit vector and $x^T x = 1$.

Consider an orthonormal basis of eigenvectors $u^1, u^2, \ldots, u^n$. Any vector $x \in R^n$ can be written as $x = \sum_{i=1}^n \alpha_i u^i$ and the expression $x^T Ax$ reduces to

$$x^T A x = \left(\sum_{i=1}^n \alpha_i u^i \right)^T A \left(\sum_{j=1}^n \alpha_j u^j \right) = \sum_{i,j=1}^n \alpha_i \alpha_j (u^i \cdot \lambda_j u^j) = \sum_{i=1}^n \alpha_i^2 \lambda_i.$$

using the fact that $u^i \cdot u^j = 1$ if $i = j$ and 0 otherwise. By a similar argument, $x^T x = \sum_{i=1}^n \alpha_i^2$ and for a unit vector we get $\sum_{i=1}^n \alpha_i^2 = 1$. I.e., the expression $x^T Ax/x^T x$ can be interpreted as a *weighted average* of the eigenvalues.

Now consider a subspace $\mathcal{U}$ generated by the first k eigenvectors, $[u^1, \ldots, u^k]$. For any unit vector $x \in \mathcal{U}$, we get $x^T Ax = \sum_{i=1}^k \alpha_i^2 \lambda_i$ and $\sum_{i=1}^k \alpha_i^2 = 1$. This weighted average is at least the smallest of the k eigenvalues, i.e.

$$\max_{dim(\mathcal{U})=k} \min_{x \in \mathcal{U}} \frac{x^T A x}{x^T x} \geq \lambda_k.$$

On the other hand, consider any subspace $\mathcal{U}$ of dimension k, and a subspace $\mathcal{V} = [u^k, u^{k+1}, \ldots, u^n]$ which has dimension $n-k+1$. These two subspace have a nontrivial intersection, namely there exists a nonzero vector $z \in \mathcal{U} \cap \mathcal{V}$. We can assume that $z = \sum_{j=k}^{n} \beta_j u^j$ is a unit vector, $z^T z = \sum_{j=k}^{n} \beta_j^2 = 1$. We obtain

$$\frac{z^T A z}{z^T z} = \sum_{j=k}^{n} \beta_j^2 \lambda_j \leq \lambda_k,$$

since this is a weighted average of the last $n-k+1$ eigenvalues and the largest of these eigenvalues is λ_k. Consequently,

$$\max_{dim(\mathcal{U})=k} \min_{x \in \mathcal{U}} \frac{x^T A x}{x^T x} \leq \frac{z^T A z}{z^T z} \leq \lambda_k.$$

□

3 A bound on the independence number

Theorem 2. *For a d-regular graph with smallest (most negative) eigenvalue λ_n, the independence number is*

$$\alpha(G) \leq \frac{n}{1 - d/\lambda_n}.$$

Keep in mind that $\lambda_n < 0$, so the denominator is larger than 1.

Proof. Let $S \subseteq V$ be a maximum independent set, $|S| = \alpha$. We consider a vector $x = n\mathbf{1}_S - \alpha\mathbf{1}$. (This vector can be seen as an indicator vector of S, modified to be orthogonal to $\mathbf{1}$.) By Lemma 1 with $\mathcal{U} = R^n$, we know that

$$\frac{x^T A x}{x^T x} \geq \lambda_n.$$

It remains to compute $x^T A x$. We get

$$x^T A x = n^2 \mathbf{1}_S^T A \mathbf{1}_S - 2\alpha n \mathbf{1}_S^T A \mathbf{1} + \alpha^2 \mathbf{1}^T A \mathbf{1}.$$

By the property of the independent set, we have $\mathbf{1}_S^T A \mathbf{1}_S = \sum_{i,j \in S} a_{ij} = 0$. Similarly, we get $\mathbf{1}_S^T A \mathbf{1} = d\mathbf{1}_S^T \cdot \mathbf{1} = \alpha d$ and $\mathbf{1}^T A \mathbf{1} = d\mathbf{1} \cdot \mathbf{1} = dn$. All in all,

$$x^T A x = -2\alpha n \cdot \alpha d + \alpha^2 \cdot dn = -\alpha^2 dn.$$

Also,

$$x^T x = n^2 ||\mathbf{1}_S||^2 - 2\alpha n \mathbf{1}_S \cdot \mathbf{1} + \alpha^2 ||\mathbf{1}||^2 = n^2 \alpha - 2\alpha^2 n + \alpha^2 n = \alpha n(n - \alpha).$$

We conclude that

$$\lambda_n \leq \frac{x^T A x}{x^T x} = \frac{-\alpha^2 dn}{\alpha n(n-\alpha)} = \frac{d}{1 - n/\alpha}$$

which implies

$$\alpha \leq \frac{n}{1 - d/\lambda_n}.$$

□

This bound need not be tight in general, but it gives the right value in many interesting cases.

- The complete graph K_n has eigenvalues $\lambda_1 = d = n - 1$ and $\lambda_n = -1$. This yields

$$\alpha(G) \leq \frac{n}{1 - d/\lambda_n} = 1.$$

- The complete bipartite graph $K_{n,n}$ has eigenvalues $\lambda_1 = d = n$ and $\lambda_n = -n$, hence

$$\alpha(G) \leq \frac{2n}{1 - d/\lambda_n} = n.$$

- The Petersen graph has eigenvalues $\lambda_1 = d = 3$ and $\lambda_n = -2$, therefore

$$\alpha(G) \leq \frac{n}{1 - d/\lambda_n} = \frac{10}{1 - 3/(-2)} = 4$$

  which is the right value.