The Petersen graph

As a more interesting exercise, we will compute the eigenvalues of the Petersen graph.

Definition 1. The Petersen graph is a graph with 10 vertices and 15 edges. It can be described in the following two ways:

1. The Kneser graph $KG(5, 2)$, of pairs on 5 elements, where edges are formed by disjoint edges.
2. The complement of the line graph of $K_5$: the vertices of the line graph are the edges of $K_5$, and two edges are joined if they share a vertex.
3. Take two disjoint copies of $C_5$: $(v_1, v_2, v_3, v_4, v_5)$ and $(w_1, w_2, w_3, w_4, w_5)$. Then add a matching of 5 edges between them: $(v_1, w_1), (v_2, w_3), (v_3, w_5), (v_4, w_2), (v_5, w_4)$.

The Petersen graph is a very interesting small graph, which provides a counterexample to many graph-theoretic statements. For example,

- It is the smallest bridgeless 3-regular graph, which has no 3-coloring of the edges so that adjacent edges get different colors (the smallest “snark”).
- It is the smallest 3-regular graph of girth 5.
- It is the largest 3-regular graph of diameter 2.
- It has 2000 spanning trees, the most of any 3-regular graph on 10 vertices.

To compute the eigenvalues of the Petersen graph, we use the fact that it is strongly regular. This means that not only does each vertex have the same degree (3), but each pair of vertices $(u, v) \in E$ has the same number of shared neighbors (0), and each pair of vertices $(u, v) \notin E$ has the same number of shared neighbors (1). In terms of the adjacency matrix, this can be expressed as follows:

- $(A^2)_{ij} = \sum_k a_{ik}a_{kj}$ is the number of neighbors shared by $i$ and $j$.
- For $i = j$, $(A^2)_{ij} = 3$.
- For $i \neq j$, $(A^2)_{ij} = 1 - a_{ij}$: either 0 or 1 depending on whether $(i, j) \in E$.

In concise form, this can be written as

$$A^2 + A - 2I = J.$$
Now consider any eigenvector, $Ax = \lambda x$. We know that one eigenvector is $1$ which has eigenvalue $d = 3$. Other than that, all eigenvectors $x$ are orthogonal to $1$, which also means that $Jx = 0$. Then we get

$$(A^2 + A - 2I)x = \lambda^2 x + \lambda x - 2x = 0.$$ 

This means that each eigenvalue apart from the largest one should satisfy a quadratic equation $\lambda^2 + \lambda - 2 = 0$. This equation has two roots, $1$ and $-2$.

Finally, we calculate the multiplicity of each root from the condition that $\sum \lambda_i = 0$. The largest eigenvalue has multiplicity 1 (it is obvious that any vector such that $Ax = 3x$ is a multiple of $1$). Therefore, if eigenvalue 1 comes with multiplicity $a$ and $-2$ with multiplicity $b$, we get $3 + a \cdot 1 + b \cdot (-2) = 0$ and $a + b = 9$, which implies $a = 5$ and $b = 4$. We conclude that the Petersen graph has eigenvalues including multiplicities $(3, 1, 1, 1, 1, -2, -2, -2, -2)$.

Finally, we show an application of eigenvalues to the following question. Consider 3 overlapping copies of the Petersen graph. The degrees in each copy are equal to 3, so the degrees in total could add up to 9 and form the complete graph $K_{10}$. However, something does not work here when you try it. The following statement shows that indeed this is impossible.

**Theorem 1.** There is no decomposition of the edge set of $K_{10}$ into 3 copies of the Petersen graph.

**Proof.** Suppose that $A, B, C$ are adjacency matrices of different permutations of the Petersen graph, such that they add up to the adjacency matrix of $K_{10}$, $A + B + C = J - I$. Let $V_A$ and $V_B$ be the subspaces corresponding to eigenvalue 1 for matrices $A$ and $B$, respectively. We know that $\dim(V_A) = \dim(V_B) = 5$, and moreover both $V_A$ and $V_B$ are orthogonal to the eigenvector $1$. This implies that they cannot be disjoint (then we would have 11 independent vectors in $\mathbb{R}^{10}$), and therefore there is a nonzero vector $z \in V_A \cap V_B$. This vector is also orthogonal to 1, i.e. $Jz = 0$. Therefore, we get $Cz = (J - I - A - B)z = -z - Az - Bz = -3z$.

But we know that $-3$ is not an eigenvalue of the Petersen graph, which is a contradiction. \hfill \Box

## 2 Moore graphs and cages

The Petersen graph is a special case of the following kind of graph: Suppose that $G$ is $d$-regular, starting from any vertex it looks like a tree up to distance $k$ and within distance $k$ we already see the entire graph. In other words, the diameter of the graph is $k$ and the girth is $2k + 1$. Such graphs are called **Moore graphs**.

By simple counting, we get that the number of vertices in such a graph must be

$$n_{d,k} = 1 + d \sum_{i=0}^{k-1} (d - 1)^i.$$ 

This is obviously the minimum possible number of vertices for a $d$-regular graph of girth $2k + 1$. Such graphs are also called **cages**.

The Petersen graph is a (unique) example of a 3-regular Moore graph of diameter 2 and girth 5. There are surprisingly few known examples of Moore graphs. We prove here that for girth 5 there cannot be too many indeed.
Theorem 2 (Hoffman-Singleton). The only $d$-regular Moore graphs of diameter 2 exist for $d = 2, 3, 7$ and possibly 57.

Proof. Assume $G$ is a $d$-regular Moore graph of girth 5. The number of vertices is $n = 1 + d + d(d - 1) = d^2 + 1$. Again, we consider the square of the adjacency matrix $A^2$. Observe that adjacent vertices don’t share any neighbors, otherwise there is a triangle in $G$. Non-adjacent vertices share exactly one neighbor, because the diameter of $G$ is 2 and there is no 4-cycle in $G$. Hence, $A^2$ has $d$ on the diagonal, 0 for edges and 1 for non-edges. In other words,

$$A^2 + A - (d - 1)I = J.$$

If $\lambda$ is an eigenvalue of $A$ different from $d$, we get $\lambda^2 + \lambda - (d - 1) = 0$. This means

$$\lambda = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4(d - 1)} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{4d - 3}.$$

Assume that $-\frac{1}{2} + \frac{1}{2}\sqrt{4d - 3}$ has multiplicity $a$ and $-\frac{1}{2} - \frac{1}{2}\sqrt{4d - 3}$ has multiplicity $b$. We get

$$d - \frac{a + b}{2} + \frac{1}{2}(a - b)\sqrt{4d - 3} = 0.$$

We also know that $a + b = n - 1 = d^2$. Therefore,

$$(a - b)\sqrt{4d - 3} = a + b - 2d = d^2 - 2d.$$

This can be true only if $a = b$ and $d = 2$, or else $4d - 3$ is a square. Let $4d - 3 = s^2$, i.e. $d = \frac{1}{4}(s^2 + 3)$. Substituting this into the equation $d - \frac{d^2}{2} + \frac{s}{2}(2a - d^2) = 0$,

we get

$$\frac{1}{4}(s^2 + 3) - \frac{1}{32}(s^2 + 3)^2 + \frac{s}{2}(2a - \frac{1}{16}(s^2 + 3)^2) = 0.$$

From here, we get

$$s^5 + s^4 + 6s^3 - 2s^2 + (9 - 32a)s = 15.$$

To satisfy this equation by integers, $s$ must divide 15 and hence $s \in \{1, 3, 5, 15\}$, giving $d \in \{1, 3, 7, 57\}$. Case $d = 1$ leads to $G = K_2$ which is not a Moore graph.

We remark that the graph for $d = 2$ is $C_5$, for $d = 3$ it is the Petersen graph, for $d = 7$ it is the “Hoffman-Singleton graph” (with 50 vertices and 175 edges) and for $d = 57$ it is not known whether such a graph exists. This graph would need to have 3250 vertices, 92,625 edges, diameter 2 and girth 5.