MAT 307: Combinatorics

Lecture 19: The Petersen graph and Moore graphs

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1 The Petersen graph

As a more interesting exercise, we will compute the eigenvalues of the *Petersen graph.*

Definition 1. *The Petersen graph is a graph with* 10 *vertices and* 15 *edges. It can be described in the following two ways:*

1. *The Kneser graph* $KG(5,2)$*, of pairs on* 5 *elements, where edges are formed by disjoint edges.*
2. *The complement of the line graph of* K_5*: the vertices of the line graph are the edges of* K_5*, and two edges are joined if they share a vertex.*
3. *Take two disjoint copies of* C_5*:* $(v_1, v_2, v_3, v_4, v_5)$ *and* $(w_1, w_2, w_3, w_4, w_5)$*. Then add a matching of* 5 *edges between them:* $(v_1, w_1), (v_2, w_3), (v_3, w_5), (v_4, w_2), (v_5, w_4)$*.*

The Petersen graph is a very interesting small graph, which provides a counterexample to many graph-theoretic statements. For example,

- It is the smallest bridgeless 3-regular graph, which has no 3-coloring of the edges so that adjacent edges get different colors (the smallest "snark").
- It is the smallest 3-regular graph of girth 5.
- It is the largest 3-regular graph of diameter 2.
- It has 2000 spanning trees, the most of any 3-regular graph on 10 vertices.

To compute the eigenvalues of the Petersen graph, we use the fact that it is *strongly regular.* This means that not only does each vertex have the same degree (3), but each pair of vertices $(u, v) \in E$ has the same number of shared neighbors (0), and each pair of vertices $(u, v) \notin E$ has the same number of shared neighbors (1). In terms of the adjacency matrix, this can be expressed as follows:

- $(A^2)_{ij} = \sum_k a_{ik} a_{kj}$ is the number of neighbors shared by i and j.
- For $i = j$, $(A^2)_{ij} = 3$.
- For $i \neq j$, $(A^2)_{ij} = 1 - a_{ij}$: either 0 or 1 depending on whether $(i, j) \in E$.

In concise form, this can be written as

$$A^2 + A - 2I = J.$$

Now consider any eigenvector, $Ax = \lambda x$. We know that one eigenvector is $\mathbf{1}$ which has eigenvalue $d = 3$. Other than that, all eigenvectors x are orthogonal to $\mathbf{1}$, which also means that $Jx = 0$. Then we get

$$(A^2 + A - 2I)x = \lambda^2 x + \lambda x - 2x = 0.$$

This means that each eigenvalue apart from the largest one should satisfy a quadratic equation $\lambda^2 + \lambda - 2 = 0$. This equation has two roots, 1 and -2.

Finally, we calculate the multiplicity of each root from the condition that $\sum \lambda_i = 0$. The largest eigenvalue has multiplicity 1 (it is obvious that any vector such that $Ax = 3x$ is a multiple of $\mathbf{1}$). Therefore, if eigenvalue 1 comes with multiplicity a and -2 with multiplicity b, we get $3 + a \cdot 1 + b \cdot (-2) = 0$ and $a + b = 9$, which implies $a = 5$ and $b = 4$. We conclude that the Petersen graph has eigenvalues including multiplicities $(3, 1, 1, 1, 1, 1, -2, -2, -2, -2)$.

Finally, we show an application of eigenvalues to the following question. Consider 3 overlapping copies of the Petersen graph. The degrees in each copy are equal to 3, so the degrees in total could add up to 9 and form the complete graph K_{10}. However, something does not work here when you try it. The following statement shows that indeed this is impossible.

Theorem 1. *There is no decomposition of the edge set of K_{10} into 3 copies of the Petersen graph.*

Proof. Suppose that A, B, C are adjacency matrices of different permutations of the Petersen graph, such that they add up to the adjacency matrix of K_{10}, $A + B + C = J - I$. Let V_A and V_B be the subspaces corresponding to eigenvalue 1 for matrices A and B, respectively. We know that $dim(V_A) = dim(V_B) = 5$, and moreover both V_A and V_B are orthogonal to the eigenvector $\mathbf{1}$. This implies that they cannot be disjoint (then we would have 11 independent vectors in R^{10}), and therefore there is a nonzero vector $z \in V_A \cap V_B$. This vector is also orthogonal to $\mathbf{1}$, i.e. $Jz = 0$. Therefore, we get

$$Cz = (J - I - A - B)z = -z - Az - Bz = -3z.$$

But we know that -3 is not an eigenvalue of the Petersen graph, which is a contradiction. □

2 Moore graphs and cages

The Petersen graph is a special case of the following kind of graph: Suppose that G is d-regular, starting from any vertex it looks like a tree up to distance k and within distance k we already see the entire graph. In other words, the diameter of the graph is k and the girth is $2k + 1$. Such graphs are called *Moore graphs*.

By simple counting, we get that the number of vertices in such a graph must be

$$n_{d,k} = 1 + d \sum_{i=0}^{k-1} (d-1)^i.$$

This is obviously the minimum possible number of vertices for a d-regular graph of girth $2k + 1$. Such graphs are also called *cages*.

The Petersen graph is a (unique) example of a 3-regular Moore graph of diameter 2 and girth 5. There are surprisingly few known examples of Moore graphs. We prove here that for girth 5 there cannot be too many indeed.

Theorem 2 (Hoffman-Singleton)**.** *The only d-regular Moore graphs of diameter* 2 *exist for* $d = 2, 3, 7$ *and possibly* 57*.*

Proof. Assume G is a d-regular Moore graph of girth 5. The number of vertices is $n = 1+d+d(d-1) = d^2+1$. Again, we consider the square of the adjacency matrix A^2. Observe that adjacent vertices don't share any neighbors, otherwise there is a triangle in G. Non-adjacent vertices share exactly one neighbor, because the diameter of G is 2 and there is no 4-cycle in G. Hence, A^2 has d on the diagonal, 0 for edges and 1 for non-edges. In other words,

$$A^2 + A - (d-1)I = J.$$

If λ is an eigenvalue of A different from d, we get $\lambda^2 + \lambda - (d-1) = 0$. This means

$$\lambda = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1+4(d-1)} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{4d-3}.$$

Assume that $-\frac{1}{2} + \frac{1}{2}\sqrt{4d-3}$ has multiplicity a and $-\frac{1}{2} - \frac{1}{2}\sqrt{4d-3}$ has multiplicity b. We get

$$d - \frac{a+b}{2} + \frac{1}{2}(a-b)\sqrt{4d-3} = 0.$$

We also know that $a + b = n - 1 = d^2$. Therefore,

$$(a-b)\sqrt{4d-3} = a + b - 2d = d^2 - 2d.$$

This can be true only if $a = b$ and $d = 2$, or else $4d-3$ is a square. Let $4d-3 = s^2$, i.e. $d = \frac{1}{4}(s^2+3)$. Substituting this into the equation

$$d - \frac{d^2}{2} + \frac{s}{2}(2a - d^2) = 0,$$

we get

$$\frac{1}{4}(s^2+3) - \frac{1}{32}(s^2+3)^2 + \frac{s}{2}(2a - \frac{1}{16}(s^2+3)^2) = 0.$$

From here, we get

$$s^5 + s^4 + 6s^3 - 2s^2 + (9-32a)s = 15.$$

To satisfy this equation by integers, s must divide 15 and hence $s \in \{1, 3, 5, 15\}$, giving $d \in \{1, 3, 7, 57\}$. Case $d = 1$ leads to $G = K_2$ which is not a Moore graph. □

We remark that the graph for $d = 2$ is C_5, for $d = 3$ it is the Petersen graph, for $d = 7$ it is the "Hoffman-Singleton graph" (with 50 vertices and 175 edges) and for $d = 57$ it is not known whether such a graph exists. This graph would need to have 3250 vertices, 92,625 edges, diameter 2 and girth 5.