MAT 307: Combinatorics

Lecture 18: Spectral graph theory

Instructor: Jacob Fox

1 Eigenvalues of graphs

Looking at a graph, we see some basic parameters: the maximum degree, the minimum degree, its connectivity, maximum clique, maximum independent set, etc. Parameters which are less obvious yet very useful are the *eigenvalues* of the graph. Eigenvalues are a standard notion in linear algebra, defined as follows.

Definition 1. *For a matrix $A \in R^{n \times n}$, a number λ is an eigenvalue if for some vector $x \neq 0$,*

$$Ax = \lambda x.$$

The vector x is called an eigenvector corresponding to λ.

Some basic properties of eigenvalues are

- The eigenvalues are exactly the numbers λ that make the matrix $A - \lambda I$ singular, i.e. solutions of $\det(A - \lambda I) = 0$.
- All eigenvectors corresponding to λ form a subspace V_λ; the dimension of V_λ is called the multiplicity of λ.
- In general, eigenvalues can be complex numbers. However, if A is a symmetric matrix ($a_{ij} = a_{ji}$), then all eigenvalues are real, and moreover there is an orthogonal basis consisting of eigenvectors.
- The sum of all eigenvalues, including multiplicities, is $\sum_{i=1}^{n} \lambda_i = Tr(A) = \sum_{i=1}^{n} a_{ii}$, the trace of A.
- The product of all eigenvalues, including multiplicities, is $\prod_{i=1}^{n} \lambda_i = \det(A)$, the determinant of A.
- The number of non-zero eigenvalues, including multiplicities, is the rank of A.

For graphs, we define eigenvalues as the eigenvalues of the *adjacency matrix*.

Definition 2. *For a graph G, the adjacency matrix $A(G)$ is defined as follows:*

- $a_{ij} = 1$ *if* $(i, j) \in E(G)$.
- $a_{ij} = 0$ *if* $i = j$ *or* $(i, j) \notin E(G)$.

Because $Tr(A(G)) = 0$, we get immediately the following.

Lemma 1. *The sum of all eigenvalues of a graph is always* 0.

Examples.

1. The complete graph K_n has an adjacency matrix equal to $A = J - I$, where J is the all-1's matrix and I is the identity. The rank of J is 1, i.e. there is one nonzero eigenvalue equal to n (with an eigenvector $\mathbf{1} = (1, 1, \ldots, 1)$). All the remaining eigenvalues are 0. Subtracting the identity shifts all eigenvalues by -1, because $Ax = (J - I)x = Jx - x$. Therefore the eigenvalues of K_n are $n - 1$ and -1 (of multiplicity $n - 1$).

2. If G is d-regular, then $\mathbf{1} = (1, 1, \ldots, 1)$ is an eigenvector. We get $A\mathbf{1} = d\mathbf{1}$, and hence d is an eigenvalue. It is easy to see that no eigenvalue can be larger than d. In general graphs, the largest eigenvalue is a certain notion of what degrees essentially are in G.

3. If G is d-regular and $d = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ are the eigenvalues of G, then the eigenvalues of $\bar{G}$ are $n - 1 - d$ and $\{-1 - \lambda_i : 2 \leq i \leq n\}$. This is because $A(\bar{G}) = J - I - A(G)$; $\bar{G}$ is $(n - 1 - d)$-regular, so the largest eigenvalue is $n - 1 - d$. Any other eigenvalue λ has an eigenvector x orthogonal to $\mathbf{1}$, and hence

$$A(\bar{G})x = (J - I - A(G))x = (0 - 1 - \lambda)x.$$

4. The complete bipartite graph $K_{m,n}$ has an adjacency matrix of rank 2, therefore we expect to have eigenvalue 0 of multiplicity $n - 2$, and two non-trivial eigenvalues. These should be equal to $\pm\lambda$, because the sum of all eigenvalues is always 0.

   We find λ by solving $Ax = \lambda x$. By symmetry, we guess that the eigenvector x should have m coordinates equal to α and n coordinates equal to β. Then,

$$Ax = (m\beta, \ldots, m\beta, n\alpha, \ldots, n\alpha).$$

   This should be a multiple of $x = (\alpha, \ldots, \alpha, \beta, \ldots, \beta)$. Therefore, we get $m\beta = \lambda\alpha$ and $n\alpha = \lambda\beta$, i.e. $mn\beta = \lambda^2\beta$ and $\lambda = \sqrt{mn}$.