MAT 307: Combinatorics

Lecture 17: Linear algebra - continued

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1 Few possible intersections - summary

Last time, we proved two results about families of sets with few possible intersection sizes. Let us compare them here.

Theorem 1. *If $\mathcal{F}$ is an L-intersecting family of subsets of $[n]$, then*

$$|\mathcal{F}| \leq \sum_{k=0}^{|L|} \binom{n}{k}.$$

Theorem 2. *Let p be prime and $L \subset Z_p$. Assume $\mathcal{F} \subset 2^{[n]}$ is an L-intersecting family (with intersections taken* mod p*), and no set in $\mathcal{F}$ has size in L (mod p). Then*

$$|\mathcal{F}| \leq \sum_{k=0}^{|L|} \binom{n}{k}.$$

Both results have intersecting applications. First, let's return to Ramsey graphs.

2 Explicit Ramsey graphs

We saw how to construct a graph on $n = \binom{k}{3}$ vertices, which does not contain any clique or independent set larger than k. Here, we improve this construction to $n = k^{\Omega(\log k/\log\log k)}$, i.e. superpolynomial in k.

Theorem 3 (Frankl,Wilson 1981)**.** *For any prime p, there is a graph G on $n = \binom{p^3}{p^2-1}$ vertices such that the size k of any clique or independent set in G is at most $\sum_{i=0}^{p-1} \binom{p^3}{i}$.*

Note that $n \simeq p^{p^2}$, while $k \simeq p^p$. I.e., $n \simeq k^p \simeq k^{\log k/\log\log k}$.

Proof. We construct G as follows. Let $V = \binom{[p^3]}{p^2-1}$, and let $A, B \in V$ form an edge if $|A \cap B| \neq p-1$ (mod p). Note that for each $A \in V$, $|A| = p^2 - 1 = p - 1$ (mod p).

If $A_1, \dots, A_k$ is a clique, then $|A_i| = p - 1$ (mod p), while $|A_i \cap A_j| \neq p - 1$ (mod p) for all $i \neq j$. By Theorem 2 with $L = \{0, 1, \dots, p-2\}$, we get $k \leq \sum_{i=0}^{p-1} \binom{p^3}{i}$.

If $A_1, \dots, A_k$ is an independent set, then $|A_i \cap A_j| = p - 1$ (mod p) for all $i \neq j$. This means $|A_i \cap A_j| \in L = \{p-1, 2p-1, \dots, p^2-p-1\}$, without any modulo operations. By Theorem 1, we get $k \leq \sum_{i=0}^{p-1} \binom{p^3}{i}$. □

3 Borsuk's conjecture

Can every bounded set $S \subset R^d$ be partitioned into $d+1$ sets of strictly smaller diameter?

This conjecture was a long-standing open problem, solved in the special cases of a sphere S (by Borsuk himself), S being a smooth convex body (using the Borsuk-Ulam theorem) and low dimension $d \leq 3$. It can be seen that a simplex requires $d+1$ sets, otherwise we have 2 vertices in the same part and hence the diameter does not decrease.

The conjecture was disproved dramatically in 1993, when Kahn and Kalai showed that significantly more than $d+1$ parts are required.

Theorem 4. *For any d sufficiently large, there exists a bounded set $S \subset R^d$ (in fact a finite set) such that any partition of S into fewer than $1.2^{\sqrt{d}}$ parts contains a part of the same diameter.*

The proof uses an algebraic construction, relying on the following lemma.

Lemma 1. *For any prime p, there exists a set of $\frac{1}{2}\binom{4p}{2p}$ vectors $F \subseteq \{-1,+1\}^{4p}$ such that every subset of $2\binom{4p}{p-1}$ vectors contains an orthogonal pair of vectors.*

Proof. Consider $4p$ elements and all subsets of size $2p$, containing a fixed element 1:

$$\mathcal{F} = \{I : I \subseteq [4p], |I| = 2p, 1 \in I\}.$$

For each set I, we define a vector $v_i^I = +1$ if $i \in I$ and $v_i^I = -1$ if $i \notin I$. We set $F = \{v^I : I \in \mathcal{F}\}$.

The only way that a pair of such vectors v^I, v^J can be orthogonal is that $|I \Delta J| = 2p$ and then $|I \cap J| = p$. Note that $|I \cap J|$ is always between 1 and $2p-1$ (I, J are different and they share at least 1 element). Hence $v^I \cdot v^J = 0$ iff $|I \cap J| = 0 \pmod{p}$.

We claim that this is the desired collection of vectors. For a subset $G \subset F$ without any orthogonal pair, we would have a family of sets $\mathcal{G} \subset \mathcal{F}$ such that

- $\forall I \in \mathcal{G}; |I| = 0 \pmod{p}$.
- $\forall$ distinct $I, J \in \mathcal{G}; |I \cap J| \in \{1, 2, \ldots, p-1\} \pmod{p}$.

By Theorem 2,

$$|\mathcal{G}| \leq \sum_{k=0}^{p-1} \binom{4p}{k} < 2\binom{4p}{p-1}.$$

□

Now we are ready to prove the theorem.

Proof. Given a set of vectors $F \subseteq R^n = R^{4p}$ provided by the lemma above, we define a set of vectors

$$X = \{v \otimes v : v \in F\} \subset R^{n^2}.$$

Here, each vector is a tensor product $w = v \otimes v$. More explicitly,

$$w_{ij} = v_i v_j, \quad 1 \leq i, j \leq n.$$

These vectors satisfy the following properties:

- $w \in \{-1,+1\}^{n^2}$; $||w|| = \sqrt{n^2} = n$.
- $w \cdot w' = (v \otimes v) \cdot (v' \otimes v') = (v \cdot v')^2 \geq 0$.
- w, w' are orthogonal if and only if v, v' are orthogonal.
- $||w - w'||^2 = ||w||^2 + ||w'||^2 - 2(w \cdot w') = 2n^2 - 2(v \cdot v')^2 \leq 2n^2$, and the pairs of maximum distance correspond to orthogonal vectors.

  By the lemma, any subset of $2\binom{4p}{p-1}$ vectors contains an orthogonal pair and so its diameter is the same as the original set. If we want to decrease the diameter, we must partition X into sets of size less than $2\binom{4p}{p-1}$, and the number of such parts is at least

$$\frac{|X|}{2\binom{4p}{p-1}} = \frac{\frac{1}{2}\binom{4p}{2p}}{2\binom{4p}{p-1}} = \frac{(3p+1)(3p)(3p-1)\cdots(2p+2)(2p+1)}{4(2p)(2p-1)\cdots(p+1)p} \geq \left(\frac{3}{2}\right)^{p-1}.$$

  The dimension of our space is $d = n^2 = (4p)^2$, and the number of parts must be at least $(3/2)^{p-1} = (3/2)^{\sqrt{d}/4-1}$. (The bound can be somewhat improved by a more careful analysis.)

$\square$